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TOKYO MATHEMATICAL BOOK SERIES
VOLUME IV

SPECTRAL THEORY

IN THE

HILBERT SPACE

BY

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1 9 5 3

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PREFACE

Spectral theory was considered first by D. Hilbert about bounded symmetric operators on a sequence space. J. von Neumann made this spectral theory take a new form. He generated this theory to unbounded symmetric operators, defining abstractly the Hilbert space. In this new style spectral theory was discussed by many mathematicians: F. Riesz, M. H. Stone, etc.

On the other hand a spectral theory was born in semi-ordered linear spaces, as an abstraction of the spectral theory in the Hilbert space. This spectral theory is constructed by means of generated Stieltjés integral, considering every spectrum as a continuous one. On the contrary there is another type of spectral theory established by the auther, which is constructed by means of generated Riemann integral in topological spaces considering every spectrum as a point one. This new spectral theory is explained precisely in Volume II of this Series, Modern Spectral Theory, Tokyo (1950), in the name of the second spectral theory. The auther has remarked in an earlier paper, Eine Spektraltheorie, Proc. Phys.-Math. Soc. Japan, 23(1941) 485-511, that this spectral theory also is applicable to the Hilbert space, but it is not well known. The purpose of this book is to make this new spectral theory revive in the Hilbert space.

In this text spectral theory is constructed firstly about elements of the Hilbert space, and as its application, we consider the spectral theory of the normal operators. Furthermore, by means of this new

spectral theory, we can discuss not only the mean ergodic theorem obtained by J. von Neumann, but the individual ergodic theorem obtained by J. D. Birkhoff, as a problem in the Hilbert space. Readers will be more interested, if they read this text, comparing Volume II of this Series.

This book was written during 1950-51 as Volume IV of Tokyo Mathematical Book Series. But financial difficulties made it impossible to publish this book. Recently I have obtained financial help from Education Ministry of Japan, and it has become possible to publish this book. I express my warmest thanks to those who have made effort to publish this book.

Sapporo, June 24, 1953

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INTRODUCTION

We can modify objects for consideration, and we obtain a notion of spaces. Let R be a space. Each object modified by R is said to be a point or an element of R. We shall define $a \in R$ or $R \ni a$ to mean that a is an element of R.

For two spaces R and S, a correspondence α, which assigns to every element $x \in R$ an element $\alpha(x) \in S$, is called a mapping of R into S, and $\alpha(x)$ the image of x by α.

Let R be a space. A mapping A of R into the space $\{\varepsilon, \bar{\varepsilon}\}$ composed of two elements ε and $\bar{\varepsilon}$, is called a set of R. For a set A of R we define $x \in A$ or $A \ni x$ to mean that ε corresponds to x by A, and $x \bar{\in} A$ or $A \bar{\ni} x$ to mean that $\bar{\varepsilon}$ correaponds to x by A. We shall say that a set A contains an element x, if $x \in A$, and that a set A does not contain an element x, if $x \bar{\in} A$. A set, which does not contains every element x of R, is called the empty set, and denoted by 0 A set, which contains every element of R, is called the whole space, and denoted by R.

For two sets A and B, we define $A \subset B$ or $B \supset A$ to mean that $x \in A$ implies $x \in B$. We shall say that A includes B, or that B is a subset of A, if $B \subset A$. With this definition, we have obviusly that $0 \subset A \subset R$ for every set A of R.

For a condition $C(x)$ about $x \in R$, we shall denote by

$$\{ x : C(x) \}$$

the set which contains x if and only if $C(x)$ is satisfied. For a set A of R, we define the complement A' of A as

$$A' = \{ x : x \bar{\in} A \}.$$

With this definition, we have obviously $A'' = A$.

Let Λ be an other space. Considering every set of R as an element, we obtain a space of sets of R. A mapping A_λ ($\lambda \in \Lambda$) of Λ into the space of sets of R is called a set system. For a set system A_λ ($\lambda \in \Lambda$) we define the union $\sum_{\lambda \in \Lambda} A_\lambda$ to mean

$$\sum_{\lambda \in \Lambda} A_\lambda = \{ x : x \in A_\lambda \text{ for some } \lambda \in \Lambda \},$$

and the intersection $\prod_{\lambda\in\Lambda} A_\lambda$ to mean

$$\prod_{\lambda\in\Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for all } \lambda \in \Lambda\}.$$

With this definition we have obviously

$$(\sum_{\lambda\in\Lambda} A_\lambda)' = \prod_{\lambda\in\Lambda} A_\lambda', \qquad (\prod_{\lambda\in\Lambda} A_\lambda)' = \sum_{\lambda\in\Lambda} A_\lambda'$$

For a set sequence A_ν $(\nu = 1, 2, \ldots)$ we may write

$$\sum_{\nu=1}^{\infty} A_\nu = A_1 \dot{+} A_2 \dot{+} \cdots, \qquad \prod_{\nu=1}^{\infty} A_\nu = A_1 A_2 \cdots .$$

For two sets A and B, if $A \supset B$, then we denote AB' by $A - B$, that is,

$$A - B = \{x : x \in A, \ x \bar{\in} B\}$$

As a method for infinite process we are permitted to make use of the following axiom due to Zermelo:

Choice Axiom. For any space R we can find a mapping of the space of sets of R into the space R such that every set A of R contains the image of A if $A \neq 0$.

By virtue of Choice Axiom we can prove the following theorem due to Zorn:

Maximal Theorem. Let $\mathfrak{C}(x_1, x_2, \ldots, x_\kappa)$ be a condition for every finite number of elements $x_\nu \in R$ $(\nu = 1, 2, \ldots, \kappa)$. This condition $\mathfrak{C}$ is said to be satisfied by a set A of R, if $\mathfrak{C}(x_1, \ldots, x_\kappa)$ is satisfied for every finite number of elements $x_\nu \in A$ $(\nu = 1, 2, \ldots, \kappa)$. If there is a set A_0 of R by which $\mathfrak{C}$ is satisfied, then we can find a maximal set A of R by which $\mathfrak{C}$ is satisfied, that is, there is no other set including A by which $\mathfrak{C}$ is satisfied. (C.f. Introduction in Volume II or III in this Series: H. Nakano, Modern spectral theory, or Topology and linear topological spaces.)

For two spaces R and S, a mapping α of R into S is said to be a one-to-one correspondence between R and S, if for every different elements $x, y \in R$, the images $\alpha(x)$ and $\alpha(y)$ are different, and

for any $y \in S$ we can find $x \in R$ for which $y = \alpha(x)$. For a one-to-one correspondence α between R and S, putting

$$x = \beta(y) \qquad \text{for } y = \alpha(x),$$

we also obtain obviously a one-to-one correspondence between S and R. This one-to-one correspondence β is called the inverse of α and denoted by α^{-1}. Concerning the inverse α^{-1}, we have obviously by definition

$$x = \alpha^{-1}(\alpha(x)) \quad \text{for } x \in R; \qquad y = \alpha(\alpha^{-1}(y)) \quad \text{for } y \in S.$$

If a set A of R is not the empty set, then we can consider A itself as a space of all elements $x \in A$. For two sets A, B different from the empty set, if there exists a one-to-one correspondence between A and B considering A and B as two spaces, then we shall say that A and B have the same density, or that the density of A coincides with that of B, and we write $A \sim B$. With this definition, we see easily that $A \sim A$ for every set $A \neq 0$; $A \sim B$ implies $B \sim A$; and $A \sim B$, $B \sim C$ implies $A \sim C$. Thus we obtain density classes, i.e. every two sets in the same density class have the same density, and every two sets in different density classes have different densities. This density classes will be called cardinal numbers. We define further by 0 the cardinal number of the empty set. Cardinal numbers may be called often densities.

For two cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$, we define $\mathfrak{m} \geqq \mathfrak{n}$ or $\mathfrak{n} \leqq \mathfrak{m}$ to mean that we can find two sets A and B such that $\mathfrak{m} \ni A \supset B \in \mathfrak{n}$. With this definition, we have obviously that $\mathfrak{m} \leqq \mathfrak{n}$, $\mathfrak{n} \leqq \mathfrak{k}$ implies $\mathfrak{m} \leqq \mathfrak{k}$, and $0 \leqq \mathfrak{k} \leqq \mathfrak{k}_R$ for every cardinal number $\mathfrak{k}$ and the cardinal number $\mathfrak{k}_R$ of R. Furthermore we have the following theorem due to Bernstein:

Theorem. $\mathfrak{m} \leqq \mathfrak{n}$, $\mathfrak{n} \leqq \mathfrak{m}$ implies $\mathfrak{m} = \mathfrak{n}$.

Proof. If $\mathfrak{m} \leqq \mathfrak{n}$, $\mathfrak{n} \leqq \mathfrak{m}$, then we can find by definition A_0, $A_1 \in \mathfrak{m}$ and $B_0 \in \mathfrak{n}$ such that $A_0 \supset B_0 \supset A_1$. Since A_0, $A_1 \in \mathfrak{m}$, there exists a one-to-one correspondence α between A_0 and A_1. For each set $X \subset A_0$, putting

$$\alpha(X) = \{\alpha(x) : \quad x \in X\}, \qquad \alpha(0) = 0,$$

we have obviously $\alpha(X) \subset A_1$. Thus, putting

$$A_\nu = \alpha(A_{\nu-1}), \quad B_\nu = \alpha(B_{\nu-1}) \qquad \text{for } \nu = 1, 2, \ldots,$$

we have $A_0 \supset B_0 \supset A_1 \supset B_1 \supset A_2 \supset B_2 \supset \cdots$, and

$$\alpha(A_{\nu-1} - B_{\nu-1}) = (A_\nu - B_\nu),$$
$$\alpha(B_{\nu-1} - A_\nu) = (B_\nu - A_{\nu+1})$$

for every $\nu = 1, 2, \ldots$. Since

$$A_0 = \prod_{\nu=1}^{\infty} A_\nu + \sum_{\nu=1}^{\infty} \{(A_{\nu-1} - B_{\nu-1}) + (B_{\nu-1} - A_\nu)\},$$
$$B_0 = \prod_{\nu=1}^{\infty} A_\nu + \sum_{\nu=1}^{\infty} \{(B_{\nu-1} - A_\nu) + (A_\nu - B_\nu)\},$$

putting

$$f(x) = \begin{cases} \alpha(x) & \text{for } x \in \sum_{\nu=1}^{\infty} (A_{\nu-1} - B_{\nu-1}), \\ x & \text{for } x \in \sum_{\nu=1}^{\infty} (B_{\nu-1} - A_\nu) + \prod_{\nu=1}^{\infty} A_\nu, \end{cases}$$

we obtain a one-to-one correspondence f between A_0 and B_0, and hence $m = n$.

If a set A of R is <u>infinite</u>, that is, if A contains an infinite number of elements, then the density of A is said to be <u>infinite</u>, or we shall say that the cardinal number of A is infinite. For a set A of R, if there exists a one-to-one correspondence between A and the totality of natural numbers, then the density of A is said to be <u>countable</u>, and the cardinal number of the countable density is denoted by $\aleph_0$.

<u>Theorem</u>. $\aleph_0$ <u>is the least cardinal number among the infinite cardinal numbers</u>.

<u>Proof</u>. Let A be a set of an infinite density, and $a_X \in X$ $(0 \neq X \subset R)$ be a mapping by Choice Axiom. For an element $a \in A$, putting

$$a_1 = a, \qquad a_\nu = a_{A - \{a_1, \ldots, a_{\nu-1}\}} \quad (\nu = 1, 2, \ldots),$$

we can find by induction a sequence $a_\nu \in A$ $(\nu = 1, 2, \ldots)$. Thus we have by definition $\aleph_0 \leqq C_A$ for the cardinal number C_A of A.

Let C_λ $(\lambda \in \Lambda)$ be a system of cardinal numbers. For a set $A_\lambda \in C_\lambda$ $(\lambda \in \Lambda)$, we obtain a space S_Λ of elements

$$(\lambda, x) \qquad \text{for } \lambda \in \Lambda,\ x \in A_\lambda.$$

If there exists a set A of R such that we can find a one-to-one correspondence between S_Λ and A, then for the cardinal number $\mathfrak{c}_A$ of A we shall write

$$\mathfrak{c}_A = \sum_{\lambda \in \Lambda} \mathfrak{c}_\lambda .$$

If Λ is a set of R with the cardinal number $\mathfrak{c}_\Lambda$ and $\mathfrak{c}_\lambda = \mathfrak{m}$ for all $\lambda \in \Lambda$, then we shall write

$$\mathfrak{c}_A = \mathfrak{c}_\Lambda \mathfrak{m}.$$

With this definition, we see easily that $\mathfrak{m}\mathfrak{n} = \mathfrak{n}\mathfrak{m}$, and $(\mathfrak{m}\mathfrak{n})\mathfrak{c} = \mathfrak{m}(\mathfrak{n}\mathfrak{c})$ for cardinal numbers $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{c}$.

Theorem. $\aleph_0 \mathfrak{c} = \mathfrak{c}$ for every infinite cardinal number $\mathfrak{c}$.

Proof. Between two spaces (ν, μ) $(\nu, \mu = 1, 2, \ldots)$ and λ $(\lambda = 1, 2, \ldots)$, putting

$$\lambda = \frac{1}{2}(\nu + \mu - 2)(\lambda + \mu - 1) + \nu \qquad (\nu, \mu = 1, 2, \ldots),$$

we obtain a one-to-one correspondence. Thus we have $\aleph_0 \aleph_0 = \aleph_0$ by definition.

If a cardinal number $\mathfrak{c}$ is infinite, then for a set $A \in \mathfrak{c}$ we can find by Maximal Theorem a system $A_\lambda \in \aleph_0$ $(\lambda \in \Lambda)$ such that

$$A = \sum_{\lambda \in \Lambda} A_\lambda, \qquad A_\lambda A_\rho = 0 \quad \text{for } \lambda \neq \rho.$$

Thus there exists a cardinal number $\mathfrak{m}$ for which $\mathfrak{c} = \aleph_0 \mathfrak{m}$, and hence

$$\aleph_0 \mathfrak{c} = \aleph_0(\aleph_0 \mathfrak{m}) = (\aleph_0 \aleph_0)\mathfrak{m} = \aleph_0 \mathfrak{m} = \mathfrak{c}.$$

A space R is said to be linearly ordered, if for every two elements x, $y \in R$ it is defined $x \leqq y$ or $y \leqq x$ such that $x \leqq y$, $y \leqq z$ implies $x \leqq z$; and $x \leqq y$, $y \leqq x$ implies $x = y$. A linearly ordered space R is said to be well ordered, if every set $A \neq 0$ of R contains the minimum element, that is, if there exists $a \in A$ such that $a \leqq x$ for every $x \in A$.

Zermelo proved by Choice Axiom that every space may be well ordered. We shall prove now however this theorem by Maximal Theorem. Every finite set may be obviously well ordered. We consider the totality of well ordered sets of R. For two well ordered sets A and B, we

shall say that A is a section of B, if $A \subset B$; $x \leqq y$ in A is equivalent to $x \leqq y$ in B for every x, $y \in A$; and $x \leqq y$ for $x \in A$, $y \in B - A$. By virtue of Maximal Theorem, we can find a maximal system $\mathfrak{M}$ of well ordered sets of R subject to the condition: for every two sets of $\mathfrak{M}$, one of them is a section of the other. Then, putting $A = \sum_{X \in \mathfrak{M}} X$, we obtain a well ordered set A such that every $X \in \mathfrak{M}$ is a section of A. Furthermore we have $A = R$. Because, if $A \neq R$, then for an element $a \in R - A$, putting $x \leqq a$ for every $x \in A$, we obtain a well ordered set $\{A, a\}$ such that every $X \in \mathfrak{M}$ is a section of $\{A, a\}$, and it is evident $\{A, a\} \bar{\in} \mathfrak{M}$, contradicting the maximality of $\mathfrak{M}$. Thus R may be well ordered.

Let R be a well ordered space. We shall prove that for any set $A \neq 0$ of R there exists a section of R which has the same density with A. We can assume obviously that the density of A is different from that of R. We consider one-to-one correspondences α_X between a section X of R and a subset of A. By virtue of Maximal Theorem, we can find a maximal system $\mathfrak{M}$ of one-to-one correspondences α_X such that for every α_{X_1}, $\alpha_{X_2} \in \mathfrak{M}$, one of X_1 and X_2 is a section of the other and $\alpha_{X_1}(x) = \alpha_{X_2}(x)$ for every $x \in X_1, X_2$. For such a maximal system $\mathfrak{M}$, putting

$$X_0 = \sum_{\alpha_X \in \mathfrak{M}} X, \qquad \alpha_{X_0}(x) = \alpha_X(x) \quad \text{for} \quad x \in X,$$

we obtain a one-to-one correspondence α_{X_0} of a section X_0 of R and a subset of A, and we have $\alpha_{X_0} \in \mathfrak{M}$ by the maximality of $\mathfrak{M}$. Since the density of A is different from that of R, we have obviously $X_0 \neq R$. If $\alpha_{X_0}(X_0) \neq A$, then for an element $a \in A - \alpha_{X_0}(X_0)$, putting

$$\alpha_{\{X_0, x_0\}}(x) = \alpha_{X_0}(x) \qquad \text{for every} \quad x \in X_0$$

and $\alpha_{\{X_0, x_0\}}(x_0) = a$ for the minimum element x_0 of $R - X_0$, we obtain a one-to-one correspondence $\alpha_{\{X_0, x_0\}}$ of a section $\{X_0, x_0\}$ of R and a subset of A, and it is obvious that $\alpha_{\{X_0, x_0\}} \bar{\in} \mathfrak{M}$, contradicting the maximality of $\mathfrak{M}$. Thus we have $A = \alpha_{X_0}(X_0)$.

For every section $X \neq R$ of R, denoting by x_X the minimum element of $R - X$, we have obviously $x_X \leqq x_Y$ if and only if $X \subset Y$. Therefore we can state

Theorem. For every space R, the totality of cardinal numbers in R is a well ordered space.

Concerning topology, we shall make use of notations and terminologies in Volume II or III in this Series. But, in this text, we need only be aquainted with elementary properties as stated in Introduction in Volume II : Modern spectral theory.

Concerning complex numbers, we shall make use of usual notations: for a complex number α, we denote by $\mathcal{R}\alpha$ the real part; by $\mathcal{J}\alpha$ the imaginary part; by $\overline{\alpha}$ the conjugate complex number; and by $|\alpha|$ the absolute value, i.e.

$$\alpha = \mathcal{R}\alpha + i\,\mathcal{J}\alpha, \qquad \overline{\alpha} = \mathcal{R}\alpha - i\,\mathcal{J}\alpha,$$
$$|\alpha| = \sqrt{(\mathcal{R}\alpha)^2 + (\mathcal{J}\alpha)^2} = \sqrt{\alpha\overline{\alpha}}$$

The space of all complex numbers is called the complex plane and denoted by G. A topology is defined on the complex plane in usual sense: the system of point sets

$$\{\zeta : \ |\zeta - \alpha| < \varepsilon\} \qquad \text{for all } \varepsilon > 0, \ \alpha \in G$$

constitutes a neighbourhood system.

Let R be a topological space. A mapping φ of R into the complex plane G is called a function on R. If the function value $\varphi(x)$ is real for every $x \in R$, then φ is said to be real valued. A function φ on R is said to be continuous at a point $x_0 \in R$, if for any $\varepsilon > 0$ we can find an open set $A \ni x_0$ such that

$$|\varphi(x) - \varphi(x_0)| < \varepsilon \qquad \text{for every } x \in A.$$

A function φ is said to be continuous in a set $A \subset R$, if φ is continuous at every point of A. We may often consider a function φ which takes the infinite point ∞ as a function value. In the case: $\varphi(x_0) = \infty$, φ is said to be continuous at x_0, if for any $\varepsilon > 0$ we can find an open set $A \ni x_0$ such that

$$|\varphi(x)| > \varepsilon \qquad \text{for every } x \in A,$$

considering $|\infty| = +\infty$.

CHAPTER I

HILBERT SPACES

§1 Linear Spaces

A space R is called a commutative group, if for every two elements $a, b \in R$ we have $a + b \in R$ such that

1) $a + b = b + a$,

2) $(a + b) + c = a + (b + c)$,

3) for any a, $b \in R$ we can find $c \in R$ such that $a = b + c$.

Such an element $c \in R$ is uniquely determined. Because, if

$$a = b + c = b + c',$$

then we can find by the postulate 3) e, $d \in R$ such that

$$c = c' + d, \qquad c' = a + e,$$

and we have by the postulates 1) and 2)

$$\begin{aligned} c' = a + e &= (b + c) + e = (b + (c' + d)) + e \\ &= ((b + c') + d) + e = (a + d) + e = (a + e) + d \\ &= c' + d = c. \end{aligned}$$

Therefore we denote such $c \in R$ by $a - b$. Thus we have

$$a = b + (a - b) \qquad \text{for every } a, b \in R.$$

Especially $b = b + (b - b)$, and hence we have by 1) and 2)

$$\begin{aligned} a &= (b + (b - b)) + (a - b) \\ &= (b + (a - b)) + (b - b) = a + (b - b). \end{aligned}$$

By the uniqueness of $a - a$ we conclude therefore

$$a - a = b - b \qquad \text{for every } a, b \in R,$$

that is, for every $a \in R$ we obtain the same element $a - a$. This uniquely determined element $a - a$ is called the zero element of R and denoted by 0.

For each element a of a commutative group R we define $-a$ to mean $0 - a$. With this definition, we have $-(-a) = a$, because

$$0 = a + (-a) = (-a) + (-(-a)).$$

Since $b + (a + (-b)) = a + (b + (-b)) = a + 0 = a$, we have

$$a + (-b) = a - b \qquad \text{for every } a, b \in R.$$

A commutative group R is called a linear space, if for every $a \in R$ and complex number α we have $\alpha a \in R$ such that

4) $\alpha(\beta a) = (\alpha\beta)a$,

5) $\alpha a + \beta a = (\alpha + \beta)a$,

6) $\alpha a + \alpha b = \alpha(a + b)$,

7) $1a = a$.

Since $0a + 0a = 0a$ by the postulate 5), we have $0a = 0$ for every $a \in R$. Therefore we have by 5) and 7)

$$a + (-1)a = 0a = 0, \text{ and hence } (-1)a = -a.$$

Furthermore $\alpha(a - b) = \alpha a - \alpha b$ for every $a, b \in R$, because

$$\alpha(a - b) = \alpha(a + (-b)) = \alpha a + \alpha(-1)b = \alpha a - \alpha b.$$

Let R be a linear space in the sequel. A set of R is called a manifold of R, if it is not the empty set. A manifold A is said to be linear, if $a, b \in A$ implies $\alpha a + \beta b \in A$ for every complex numbers α, β.

For a manifold A of R, an element $a \in R$ is said to be a linear combination from A, if we can find a finite number of elements $x_\nu \in A$ and complex numbers α_ν $(\nu = 1, 2, \ldots, \kappa)$ such that

$$a = \sum_{\nu=1}^{\kappa} \alpha_\nu x_\nu .$$

We see easily that the totality of linear combinations from A constitutes a linear manifold of R. This linear manifold of R is called the linear manifold generated by A. It is evident by definition that the linear manifold generated by A is the least linear manifold including A.

A system $a_\lambda \in R$ $(\lambda \in \Lambda)$ is said to be linearly independent, if every one is not a linear combination from the others, that is, if for a finite number of elements $\lambda_\nu \in \Lambda$ and complex numbers α_ν $(\nu = 1, 2, \ldots, \kappa)$

$$\sum_{\nu=1}^{\kappa} \alpha_\nu a_{\lambda_\nu} = 0 \quad \text{implies} \quad \alpha_\nu = 0 \quad \text{for every} \quad \nu = 1, 2, \ldots, \kappa.$$

Functions on a linear space R are called functionals on R. A functional φ on R is said to be linear, if

$$\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$$

for every elements x, $y \in R$ and complex numbers α, β.

For two functionals φ, ψ on R and complex numbers α, β, we define a functional $\alpha\varphi + \beta\psi$ as

$$(\alpha\varphi + \beta\psi)(x) = \alpha\varphi(x) + \beta\psi(x) \qquad \text{for } x \in R.$$

With this definition, we see easily that the totality of linear functionals on R constitutes a linear space.

A functional φ on R is said to be anti-linear, if

$$\varphi(\alpha x + \beta y) = \bar{\alpha}\varphi(x) + \bar{\beta}\varphi(y)$$

for every elements x, $y \in R$ and complex numbers α, β, that is, φ is anti-linear, if and only if, putting $\bar{\varphi}(x) = \overline{\varphi(x)}$ for every $x \in R$, we obtain a linear functional $\bar{\varphi}$ on R. With this definition, we also see easily that the totality of anti-linear functionals on R constitutes a linear space.

§2 Normed Spaces

Let R be a linear space. A functional $\|x\|$ ($x \in R$) is called a norm on R, if

1) $0 \leqq \|x\| < +\infty$ for every $x \in R$,

2) $\|\alpha x\| = |\alpha| \|x\|$ for every complex number α,

3) $\|x + y\| \leqq \|x\| + \|y\|$,

4) $\|x\| = 0$ implies $x = 0$.

A linear space associated with a norm is called a normed space.

Let R be a normed space in the sequel. A sequence $a_\nu \in R$ ($\nu = 1, 2, \ldots$) is said to be convergent to a limit $a \in R$ and we shall write

$$\lim_{\nu\to\infty} a_\nu = a,$$

if $\lim_{\nu\to\infty} \|a_\nu - a\| = 0$. If a sequence $a_\nu \in R$ ($\nu = 1, 2, \ldots$) is convergent, then its limit is determined uniquely. Because, if

$$\lim_{\nu\to\infty} \|a_\nu - a\| = \lim_{\nu\to\infty} \|a_\nu - b\| = 0,$$

then we conclude $\|a - b\| = 0$ by 2), 3), and hence $a = b$ by 4).

Concerning convergence, we can prove easily

Theorem 2.1. If $\lim_{\nu\to\infty} a_\nu = a$, $\lim_{\nu\to\infty} b_\nu = b$, then we have

$$\lim_{\nu\to\infty}(\alpha a_\nu + \beta b_\nu) = \alpha a + \beta b$$

for every complex numbers α, β. If $\lim_{\nu\to\infty} a_\nu = a$, $\lim_{\nu\to\infty}\alpha_\nu = \alpha$, then we have $\lim_{\nu\to\infty}\alpha_\nu a_\nu = \alpha a$.

From the postulates 2) and 3), we conclude at once

$$|\,\|x\| - \|y\|\,| \leqq \|x - y\|.$$

Therefore we have

Theorem 2.2. $\lim_{\nu\to\infty} a_\nu = a$ implies $\lim_{\nu\to\infty}\|a_\nu\| = \|a\|$.

A manifold A of R is said to be closed, if $\lim_{\nu\to\infty} a_\nu = a$, $a_\nu \in A$ $(\nu = 1, 2, \ldots)$ implies $a \in A$. With this definition, we see easily that for any two closed manifolds A, B the union $A + B$ also is closed, and that for any system of closed manifolds A_λ $(\lambda \in \Lambda)$, the intersection $\prod_{\lambda\in\Lambda} A_\lambda$ also is closed, if it is not empty.

Theorem 2.3. For a closed linear manifold A of R, if $a \bar{\in} A$, and $x_\nu + \alpha_\nu a$ $(\nu = 1, 2, \ldots)$ is convergent for a sequence of elements and complex numbers $x_\nu \in A$, α_ν $(\nu = 1, 2, \ldots)$, then both sequences x_ν and α_ν $(\nu = 1, 2, \ldots)$ are convergent.

Proof. If $\lim_{\nu\to\infty}(x_\nu + \alpha_\nu a) = b$, $x_\nu \in A$ $(\nu = 1, 2, \ldots)$, then α_ν $(\nu = 1, 2, \ldots)$ is bounded. Because, if $\lim_{\mu\to\infty}|\alpha_{\nu_\mu}| = +\infty$ for a subsequence α_{ν_μ} $(\mu = 1, 2, \ldots)$, then we have by Theorem 2.1

$$\lim_{\mu\to\infty}\|\frac{1}{\alpha_{\nu_\mu}} x_{\nu_\mu} + a\| = 0, \quad -\frac{1}{\alpha_{\nu_\mu}} x_{\nu_\mu} \in A \quad (\mu = 1, 2, \ldots),$$

contradicting the assumption: $a \bar{\in} A$ and A is closed. Therefore every subsequence from α_ν $(\nu = 1, 2, \ldots)$ contains a convergent subsequence. Furthermore, for every two convergent subsequences α_{μ_ν}, α_{σ_ν} $(\nu = 1, 2, \ldots)$, putting

$$\alpha = \lim_{\nu\to\infty}\alpha_{\mu_\nu}, \qquad \beta = \lim_{\nu\to\infty}\alpha_{\sigma_\nu},$$

we have by Theorem 2.1

$$\lim_{\nu\to\infty} x_{\mu_\nu} = b - \alpha a, \qquad \lim_{\nu\to\infty} x_{\sigma_\nu} = b - \beta a.$$

Since A is a closed linear manifold by assumption, we obtain hence

$$A \ni \lim_{\nu\to\infty} x_{\mu_\nu} - \lim_{\nu\to\infty} x_{\sigma_\nu} = (\beta - \alpha)\, a,$$

and consequently $\alpha = \beta$, because $a \bar{\in} A$. Thus we conclude that α_ν $(\nu = 1, 2, \ldots)$ is convergent, and hence $x_\nu \in A$ $(\nu = 1, 2, \ldots)$ is convergent by Theorem 2.1.

From this Theorem 2.3 we conclude immediately

Theorem 2.4. If a linear manifold A is closed, then for any $a \bar{\in} A$, the linear manifold generated by $\{a, A\}$ is closed.

We see easily by definition that the intersection of a system of closed linear manifolds also is a closed linear manifold. Therefore for any manifold A there exists the least closed linear manifold including A. This least closed linear manifold is called the closed linear manifold generated by A.

For two manifolds $B \subset A$ of R, we shall say that B is dense in A, if for any $a \in A$ we can find a sequence $b_\nu \in B$ $(\nu = 1, 2, \ldots)$ such that $\lim_{\nu \to \infty} b_\nu = a$. With this definition, we have

Theorem 2.5. The linear manifold generated by a manifold A is dense in the closed linear manifold generated by A.

Proof. Let B the linear manifold generated by A, and C the closed linear manifold generated by A. It is evident by definition that $B \subset C$. Let C_0 be the totality of elements $x \in R$ subject to the condition: $\lim_{\nu \to \infty} b_\nu = x$ for some sequence $b_\nu \in B$ $(\nu = 1, 2, \ldots)$. Then we see easily by Theorem 2.1 that C_0 is a linear manifold including B. Furthermore we see easily by definition that C_0 is closed. Since C_0 must be included in C by definition, we conclude hence that $C_0 = C$.

A sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ is said to be a Cauchy sequence, if for any $\varepsilon > 0$ we can find ν_0 such that

$$\| a_\nu - a_\mu \| \leqq \varepsilon \qquad \text{for} \quad \nu, \mu \geqq \nu_0,$$

that is, if $\lim_{\nu, \mu \to \infty} \| a_\nu - a_\mu \| = 0$. It is evident by definition, that every convergent sequence of elements is a Cauchy sequence. A manifold A is said to be complete, if every Cauchy sequence $a_\nu \in A$ $(\nu = 1, 2, \ldots)$ is convergent to a limit contained in A.

With this definition, we have obviously

Theorem 2.6. Every complete manifold is closed.

Furthermore we see easily by definition

Theorem 2.7. For a complete manifold A and a closed manifold

B, the intersection AB is a complete manifold, if it is not empty.

Theorem 2.8. If a linear manifold A is complete, then for any $a \bar{\in} A$ the linear manifold generated by $\{a, A\}$ is complete.

Proof. For $x_\nu \in A$ $(\nu = 1, 2, \ldots)$ and $a \bar{\in} A$, if $x_\nu + \alpha_\nu a$ $(\nu = 1, 2, \ldots)$ is a Cauchy sequence, then we have by definition

$$\lim_{\nu, \mu \to \infty} \{(x_\nu - x_\mu) + (\alpha_\nu - \alpha_\mu) a\} = 0.$$

Since A is closed by Theorem 2.6, we conclude from $a \bar{\in} A$ by Theorem 2.3 that

$$\lim_{\nu, \mu \to \infty} (x_\nu - x_\mu) = 0, \qquad \lim_{\nu, \mu \to \infty} (\alpha_\nu - \alpha_\mu) = 0.$$

Since A is complete by assumption, we see that both $x_\nu \in A$ and α_ν are convergent, and there exists $x \in A$ for which $\lim_{\nu \to \infty} x_\nu = x$.

Thus we have by Theorem 2.1

$$\lim_{\nu \to \infty} (x_\nu + \alpha_\nu a) = x + (\lim_{\nu \to \infty} \alpha_\nu) a.$$

From this Theorem 2.8 we conclude immediately

Theorem 2.9. The linear manifold generated by a finite number of elements is complete.

For a sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$, we shall say that a series $\sum_{\nu=1}^{\infty} a_\nu$ is convergent with sum a, and write

$$a = \sum_{\nu=1}^{\infty} a_\nu,$$

if $\lim_{\nu \to \infty} \sum_{\mu=1}^{\nu} a_\mu = a$. With this definition, we see easily by Theorem 2.2

$$\| \sum_{\nu=1}^{\infty} a_\nu \| \leqq \sum_{\nu=1}^{\infty} \| a_\nu \|,$$

if $\sum_{\nu=1}^{\infty} a_\nu$ is convergent.

Theorem 2.10. If a normed space R is complete and

$$\sum_{\nu=1}^{\infty} \| a_\nu \| < +\infty,$$

then $\sum_{\nu=1}^{\infty} a_\nu$ is convergent.

Proof. If $\sum_{\nu=1}^{\infty} \| a_\nu \| < +\infty$, then we have

$$\lim_{\mu, \rho \to \infty} \sum_{\nu=\mu}^{\rho} \| a_\nu \| = 0.$$

Since $\| \sum_{\nu=\mu}^{\rho} a_\nu \| \leqq \sum_{\nu=\mu}^{\rho} \| a_\nu \|$, we obtain further $\lim_{\mu, \rho \to \infty} \| \sum_{\nu=\mu}^{\rho} a_\nu \| = 0$.

Since R is complete by assumption, $\sum_{\nu=1}^{\infty} a_\nu$ is hence convergent.

§3 Adjoint Spaces

Let R be a normed space. An anti-linear functional φ on R is said to be bounded, if

$$\sup_{\|x\| \leqq 1} |\varphi(x)| < +\infty .$$

Denoting by $\overline{R}$ the totality of bounded anti-linear functionals on R, we see easily that $\overline{R}$ constitutes a linear space. Furthermore, putting

$$\|\bar{a}\| = \sup_{\|x\| \leqq 1} |\bar{a}(x)| \qquad (\bar{a} \in \overline{R}), \tag{1}$$

we can prove easily that $\overline{R}$ also is a normed space. This normed space $\overline{R}$ is called the adjoint space of R.

Concerning the norm of the adjoint space $\overline{R}$ of R, we have obviously by the definition (1)

$$\|\bar{a}\| = \sup_{0 \neq x \in R} \frac{|\bar{a}(x)|}{\|x\|} \qquad (\bar{a} \in \overline{R}), \tag{2}$$

and hence we have

$$|\bar{a}(x)| \leqq \|\bar{a}\| \|x\| \qquad \text{for } \bar{a} \in \overline{R},\ x \in R. \tag{3}$$

Every bounded anti-linear functional φ is continuous, i.e.

$$\lim_{\nu \to \infty} x_\nu = x \quad \text{implies} \quad \lim_{\nu \to \infty} \varphi(x_\nu) = \varphi(x),$$

because we have by the formula (3)

$$|\varphi(x_\nu) - \varphi(x)| \leqq \|\varphi\| \|x_\nu - x\| \qquad (\nu = 1, 2, \ldots).$$

Conversely, if an anti-linear functional φ is continuous, then φ is bounded. Because, if φ is not bounded, then we can find by definition a sequence $x_\nu \in R$ $(\nu = 1, 2, \ldots)$ such that

$$\|x_\nu\| \leqq 1, \qquad |\varphi(x_\nu)| \geqq 2^\nu \qquad (\nu = 1, 2, \ldots),$$

and then we have

$$\lim_{\nu \to \infty} \frac{1}{2^\nu} x_\nu = 0, \qquad \left|\varphi\left(\frac{1}{2^\nu} x_\nu\right)\right| \geqq 1 \quad (\nu = 1, 2, \ldots),$$

contradicting that φ is continuous.

Theorem 3.1. The adjoint space $\overline{R}$ of R is complete.

Proof. If $\lim_{\mu,\nu \to \infty} \|\bar{a}_\mu - \bar{a}_\nu\| = 0$ for a sequence $\bar{a}_\nu \in \overline{R}$ $(\nu = 1, 2, \ldots)$, then we have by the formula (3)

$$\lim_{\mu,\nu \to \infty} |\bar{a}_\mu(x) - \bar{a}_\nu(x)| = 0 \qquad \text{for every } x \in R.$$

Thus, putting

$$\varphi(x) = \lim_{\nu \to \infty} \bar{a}_\nu(x) \qquad (x \in R),$$

we see easily that φ is an anti-linear functional on R. Furthermore we obtain by (3) for every $\mu = 1, 2, \ldots$

$$|\bar{a}_\mu(x) - \varphi(x)| \leqq \|x\| \varlimsup_{\nu \to \infty} \|\bar{a}_\mu - \bar{a}_\nu\| \qquad \text{for every } x \in R.$$

From this relation we conclude that φ is bounded, that is, $\varphi \in \bar{R}$, and we have

$$\|\bar{a}_\mu - \varphi\| \leqq \varlimsup_{\nu \to \infty} \|\bar{a}_\mu - \bar{a}_\nu\| \qquad (\mu = 1, 2, \ldots).$$

This relation yields

$$\lim_{\mu \to \infty} \|\bar{a}_\mu - \varphi\| = 0.$$

Therefore $\bar{R}$ is complete by definition.

Theorem 3.2. *For a system of bounded anti-linear functionals* $\bar{a}_\lambda \in \bar{R}$ $(\lambda \in \Lambda)$ *on* R, *if*

$$\sup_{\lambda \in \Lambda} |\bar{a}_\lambda(x)| < +\infty \qquad \text{for every } x \in R,$$

and R *is complete, then we have*

$$\sup_{\lambda \in \Lambda} \|\bar{a}_\lambda\| < +\infty.$$

Proof. If $\sup_{\lambda \in \Lambda} \|\bar{a}_\lambda\| = +\infty$, then we can find by definition $\lambda_\nu \in \Lambda$ and $x_\nu \in R$ $(\nu = 1, 2, \ldots)$ such that

$$\|x_\nu\| = 1, \qquad |\bar{a}_{\lambda_\nu}(x_\nu)| \geqq 2^{2\nu} \qquad (\nu = 1, 2, \ldots).$$

Then, putting

$$\bar{b}_\nu = \frac{1}{2^\nu} \bar{a}_\nu, \qquad a_\nu = \frac{1}{2^\nu} x_\nu \qquad (\nu = 1, 2, \ldots),$$

we have

$$|\bar{b}_\nu(a_\nu)| \geqq 1, \qquad \|a_\nu\| = \frac{1}{2^\nu} \qquad (\nu = 1, 2, \ldots),$$

$$\lim_{\nu \to \infty} \bar{b}_\nu(x) = 0 \qquad \text{for every } x \in R,$$

$$\lim_{\mu \to \infty} \bar{b}_\nu(a_\mu) = 0 \qquad \text{for every } \nu = 1, 2, \ldots.$$

Therefore we can find a subsequence ν_μ $(\mu = 1, 2, \ldots)$ such that

$$|\bar{b}_{\nu_\mu}(a_{\nu_1} + \ldots + a_{\nu_{\mu-1}})| < \frac{1}{3},$$

$$|\bar{b}_{\nu_\rho}(a_{\nu_\rho})| \leqq \frac{1}{3} \frac{1}{2^\mu} \qquad \text{for } \rho < \mu.$$

Then we have obviously $\sum_{\mu=1}^{\infty} \|a_{\nu_\mu}\| \leqq \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} = 1$, and hence we can put

$$a = \sum_{\mu=1}^{\infty} a_{\nu_\mu},$$

because R is complete by assumption. For such $a \in R$ we have

$$|\bar{b}_{\nu_\mu}(a)| \geqq |\bar{b}_{\nu_\mu}(a_{\nu_\mu})| - |\bar{b}_{\nu_\mu}(a_{\nu_1} + \ldots + a_{\nu_{\mu-1}})| - |\bar{b}_{\nu_\mu}(\sum_{\rho=\mu+1}^{\infty} a_{\nu_\rho})|$$

$$\geq 1 - \frac{1}{3} - \sum_{\rho=\mu+1}^{\infty} |\bar{l}_{\nu_\mu}(a_{\nu_\rho})|$$
$$\geqq 1 - \frac{1}{3} - \sum_{\rho=\mu+1}^{\infty} \frac{1}{3}\frac{1}{2^\rho} \geq \frac{1}{3}$$

for every $\mu = 1, 2, \ldots$, contradicting $\lim_{\mu\to\infty} |\bar{l}_{\nu_\mu}(a)| = 0$.

A sequence $\bar{a}_\nu \in \bar{R}$ $(\nu = 1, 2, \ldots)$ is said to be <u>weakly convergent</u> to a <u>limit</u> $\bar{a} \in \bar{R}$, if

$$\lim_{\nu\to\infty} \bar{a}_\nu(x) = \bar{a}(x) \qquad \text{for every } x \in R,$$

and then we shall write

$$w\text{-}\lim_{\nu\to\infty} \bar{a}_\nu = \bar{a}.$$

Since $|\bar{a}_\nu(x) - \bar{a}(x)| \leqq \|\bar{a}_\nu - \bar{a}\|\,\|x\|$ by (3), we see that

$$\lim_{\nu\to\infty} \bar{a}_\nu = \bar{a} \quad \text{implies} \quad w\text{-}\lim_{\nu\to\infty} \bar{a}_\nu = \bar{a}.$$

Furthermore we conclude likewise

<u>Theorem</u> 3.3. $\lim_{\nu\to\infty} \bar{a}_\nu = \bar{a}$, $\lim_{\nu\to\infty} a_\nu = a$ <u>implies</u>

$$\lim_{\nu\to\infty} \bar{a}_\nu(a_\nu) = \bar{a}(a).$$

<u>Theorem</u> 3.4. $w\text{-}\lim_{\nu\to\infty} \bar{a}_\nu = a$ <u>implies</u> $\lim_{\nu\to\infty} \|\bar{a}_\nu\| \geqq \|\bar{a}\|$.

<u>Proof</u>. For every $x \in R$ we conclude from assumption by (3)

$$|\bar{a}(x)| = \lim_{\nu\to\infty} |\bar{a}_\nu(x)| \leqq \lim_{\nu\to\infty} \|\bar{a}_\nu\|\,\|x\|.$$

Thus we obtain by definition $\|\bar{a}\| \leqq \lim_{\nu\to\infty} \|\bar{a}_\nu\|$.

<u>Theorem</u> 3.5. <u>If R is complete, then</u>

$$w\text{-}\lim_{\nu\to\infty} \bar{a}_\nu = \bar{a}, \qquad \lim_{\nu\to\infty} a_\nu = a$$

<u>implies</u> $\lim_{\nu\to\infty} \bar{a}_\nu(a_\nu) = \bar{a}(a)$.

<u>Proof</u>. If R is complete, then we see by Theorem 3.2 that

$$w\text{-}\lim_{\nu\to\infty} \bar{a}_\nu = \bar{a}$$

implies $\sup_{\nu\geq 1} \|\bar{a}_\nu\| < +\infty$. Since

$$|\bar{a}_\nu(a_\nu) - \bar{a}(a)| \leqq |\bar{a}_\nu(a_\nu - a)| + |\bar{a}_\nu(a) - \bar{a}(a)|$$
$$\leqq \|a_\nu - a\|\,\|\bar{a}_\nu\| + |\bar{a}_\nu(a) - \bar{a}(a)|,$$

we conclude hence that $w\text{-}\lim_{\nu\to\infty} \bar{a}_\nu = \bar{a}$, $\lim_{\nu\to\infty} a_\nu = a$ implies

$$\lim_{\nu\to\infty} \bar{a}_\nu(a_\nu) = \bar{a}(a).$$

A manifold A of R is said to be <u>fundamental</u>, if the closed linear manifold generated by A coincides with the whole space R. With this definition, it is evident that if a manifold A is dense in R, then A is fundamental.

Theorem 3.6. For a fundamental manifold A of R and a sequence $\bar{a}_\nu \in \bar{R}$ $(\nu = 1, 2, \ldots)$, if $\bar{a}_\nu(x)$ $(\nu = 1, 2, \ldots)$ is convergent for every $x \in A$ and $\sup_{\nu \geq 1} \| \bar{a}_\nu \| < +\infty$, then $\bar{a}_\nu$ $(\nu = 1, 2, \ldots)$ is weakly convergent.

Proof. Denoting by B the linear manifold generated by A, we see easily by assumption that $\bar{a}_\nu(x)$ $(\nu = 1, 2, \ldots)$ is convergent for every $x \in B$. Since A is fundamental by assumption, B is dense in R by Theorem 2.5. Thus for any $x \in R$ we can find a sequence $x_\mu \in B$ $(\mu = 1, 2, \ldots)$ such that $\lim_{\mu \to \infty} x_\mu = x$. Then we have

$$\begin{aligned} |\bar{a}_\nu(x) - \bar{a}_\sigma(x)| &\leq |\bar{a}_\nu(x - x_\mu)| + |\bar{a}_\sigma(x - x_\mu)| \\ &\quad + |\bar{a}_\nu(x_\mu) - \bar{a}_\sigma(x_\mu)| \\ &\leq (\|\bar{a}_\nu\| + \|\bar{a}_\sigma\|)\|x - x_\mu\| + |\bar{a}_\nu(x_\mu) - \bar{a}_\sigma(x_\mu)| \end{aligned}$$

for every $\nu, \sigma = 1, 2, \ldots$. Since we have by assumption

$$\lim_{\nu, \sigma \to \infty} |\bar{a}_\nu(x_\mu) - \bar{a}_\sigma(x_\mu)| = 0 \qquad \text{for every } \mu = 1, 2, \ldots,$$

putting $\gamma = \sup_{\nu \geq 1} \|\bar{a}_\nu\|$, we obtain thus,

$$\overline{\lim_{\nu, \sigma \to \infty}} |\bar{a}_\nu(x) - \bar{a}_\sigma(x)| \leq 2\gamma \|x - x_\mu\|$$

for every $\mu = 1, 2, \ldots$. Making μ tend to ∞, we conclude hence

$$\lim_{\nu, \sigma \to \infty} |\bar{a}_\nu(x) - \bar{a}_\sigma(x)| = 0,$$

and consequently $\bar{a}_\nu(x)$ $(\nu = 1, 2, \ldots)$ is convergent for every $x \in R$. Therefore, putting

$$\varphi(x) = \lim_{\nu \to \infty} \bar{a}_\nu(x) \qquad (x \in R),$$

we see easily that φ is an anti-linear functional on R, and

$$\|\varphi\| = \sup_{\|x\| \leq 1} |\varphi(x)| \leq \gamma.$$

Thus $\bar{a}_\nu$ $(\nu = 1, 2, \ldots)$ is weakly convergent.

§4 Inner Product Spaces

Let R be a linear space. A functional (x, y) $(x, y \in R)$ is called an inner product on R, if

1) $(x, y) = \overline{(y, x)}$ for every $x, y \in R$,

2) $(x + y, z) = (x, z) + (y, z)$,

3) $(\alpha x, y) = \alpha(x, y)$ for every complex number α,

4) $(x, x) \geq 0$ for every $x \in R$,

5) $(x, x) = 0$ implies $x = 0$

Concerning inner products, we conclude from 1) and 2)

2') $(x, y + z) = (x, y) + (x, z)$,

and from 1) and 3)

3') $(x, \alpha y) = \bar{\alpha}(x, y)$ for every complex number α.

A linear space associated with an inner product is called an inner product space.

Let R be an inner product space in the sequel. By virtue of the postulate 4), putting

(1) $\|x\| = \sqrt{(x, x)}$ $(x \in R)$,

we have $0 \leqq \|x\| < +\infty$, and we obtain by 5) that $\|x\| = 0$ implies $x = 0$.

For $0 \neq y \in R$, putting $\lambda = -\dfrac{(x, y)}{\|y\|^2}$, we have for every $x \in R$

$$(x + \lambda y, x + \lambda y) = \|x\|^2 + 2\Re\bar{\lambda}(x, y) + |\lambda|^2\|y\|^2$$
$$= \|x\|^2 - \frac{1}{\|y\|^2}|(x, y)|^2.$$

Thus we conclude by the postulate 4)

(2) $|(x, y)| \leqq \|x\|\|y\|$ for every $x, y \in R$.

By virtue of this formula, we obtain

$$\|x + y\|^2 = \|x\|^2 + 2\Re(x, y) + \|y\|^2 \leqq (\|x\| + \|y\|)^2,$$

and hence

$$\|x + y\| \leqq \|x\| + \|y\|.$$

Therefore $\|x\|$ $(x \in R)$ is a norm on R. This norm is defined as the norm of an inner product space R. Concerning the norm of R, we have obviously by definition

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\Re(x, y) + \|y\|^2.$$

Consequently we obtain

(3) $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

Furthermore we have

$$\Re(x, y) = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2$$

From this relation we conclude

$$\Im(x, y) = \Re(x, iy) = \frac{1}{4}\|x + iy\|^2 - \frac{1}{4}\|x - iy\|^2.$$

Therefore we obtain

$$(4) \qquad (x,y) = \frac{1}{4}\{(\|x+y\|^2-\|x-y\|^2) + i(\|x+iy\|^2-\|x-iy\|^2)\}.$$

Every inner product space R is a normed space by the norm of R. Conversely we have

Theorem 4.1. For a normed space R, if the norm of R satisfies the relation (3), then there exists uniquely an inner product on R: (x,y) $(x, y \in R)$ such that

$$\|x\| = \sqrt{(x,x)} \qquad \text{for every } x \in R.$$

Proof. For the norm $\|x\|$ of R, putting

$$(x,y) = \frac{1}{4}\{(\|x+y\|^2-\|x-y\|^2) + i(\|x+iy\|^2-\|x-iy\|^2)\},$$

we have obviously

$$(x,x) = \|x\|^2 \qquad \text{for every } x \in R.$$

Furthermore we see easily that

$$(ix,y) = i(x,y), \qquad (x,iy) = -i(x,y).$$

For an arbitrary complex number λ, we have by the relation (3)

$$\begin{aligned}(\lambda x,y) + (x,\lambda y) &= \frac{1}{8}(|\lambda+1|^2-|\lambda-1|^2)\{\|x+y\|^2-\|x-y\|^2+i(\|x+iy\|^2-\|x-iy\|^2)\}\\ &= (\lambda+\bar{\lambda})(x,y).\end{aligned}$$

Therefore, putting $\lambda = i\alpha$ for an arbitrary real number α, we obtain

$$(i\alpha x,y) + (x,i\alpha y) = 0,$$

and hence $(\alpha x,y) = (x,\alpha y)$ for every real number α. Consequently, putting $\lambda = \alpha$ for an arbitrary real number α, we obtain

$$2(\alpha x,y) = (\alpha x,y) + (x,\alpha y) = 2\alpha(x,y),$$

and hence $(\alpha x,y) = \alpha(x,y)$ for every real number α. Thus we conclude now that

$$(\lambda x,y) = \lambda(x,y) \qquad \text{for every complex number } \lambda.$$

By virtue of the relation (3) we conclude further

$$(x,z) + (y,z) = \frac{1}{2}(x+y,\,2z) = (x+y,\,z).$$

Therefore (x,y) $(x, y \in R)$ is an inner product on R such that

$$(x,x) = \|x\|^2 \qquad \text{for every } x \in R.$$

The uniqueness of such an inner product is evident by the formula (4).

Every inner product space R may be considered as a normed space by the norm of R. Therefore we can consider convergence of ele-

ments sequences in an inner product space R. We see easily by the formula (2)

Theorem 4.2. $\lim_{\nu\to\infty} a_\nu = a$, $\lim_{\nu\to\infty} b_\nu = b$ implies

$$\lim_{\nu\to\infty} (a_\nu, b_\nu) = (a, b).$$

An inner product space is called a Hilbert space, if it is complete as a normed space.

§5 Adjoint Spaces of Inner Product Spaces

Let R be an inner product space and $\overline{R}$ the adjoint space of R, considering R as a normed space. For every $a \in R$, putting

(1) $$\varphi_a(x) = (a, x) \qquad (x \in R),$$

we obtain obviously an anti-linear functional φ_a on R. Recalling the formula §4(2), we see at once

(2) $$\|\varphi_a\| = \|a\| \qquad \text{for every } a \in R.$$

Therefore φ_a is contained in $\overline{R}$. Furthermore we have obviously by definition

(3) $$\varphi_{\alpha a+\beta b} = \alpha\varphi_a + \beta\varphi_b$$

for every $a, b \in R$ and complex numbers α, β.

Theorem 5.1. For a closed linear manifold A of R, if $a \overline{\in} A$, then we can find $\bar{a} \in \overline{R}$ such that

$$\bar{a}(a) \neq 0, \qquad \bar{a}(x) = 0 \quad \text{for every } x \in A.$$

Proof. Putting $\gamma = \inf_{x\in A} \|x - a\|$, we have $\gamma > 0$, because A is closed and $a \overline{\in} A$ by assumption. We can find obviously a sequence $x_\nu \in A$ $(\nu = 1, 2, \ldots)$ such that

$$\lim_{\nu\to\infty} \|x_\nu - a\| = \gamma.$$

Then we can find further a sequence $\varepsilon_\nu > 0$ $(\nu = 1, 2, \ldots)$ such that

$$\lim_{\nu\to\infty} \varepsilon_\nu = 0, \quad \|x_\nu - a\|^2 - \gamma^2 \leqq \varepsilon_\nu^2 \quad (\nu = 1, 2, \ldots).$$

For an arbitrary sequence $y_\nu \in A$ $(\nu = 1, 2, \ldots)$ such that

$$\sup_{\nu\geqq 1} \|y_\nu\| < +\infty,$$

putting

$$\alpha_\nu = -\varepsilon_\nu (x_\nu - a, y_\nu) \qquad (\nu = 1, 2, \ldots),$$

we have

$$\|\alpha_\nu y_\nu + (x_\nu - a)\|^2 = |\alpha_\nu|^2\|y_\nu\|^2 - 2\Re\bar{\alpha}_\nu(x_\nu - a, y_\nu) + \|x_\nu - a\|^2$$
$$\leqq \varepsilon_\nu^2\|x_\nu - a\|^2\|y_\nu\|^2 - 2\varepsilon_\nu|(x_\nu - a, y_\nu)|^2 + \|x_\nu - a\|^2$$
$$\leqq \varepsilon_\nu^2(\varepsilon_\nu^2 + \gamma^2)\|y_\nu\|^2 - 2\varepsilon_\nu|(x_\nu - a, y_\nu)|^2 + \varepsilon_\nu^2 + \gamma^2.$$

On the other hand, we have naturally $\alpha_\nu y_\nu + x_\nu \in A$ $(\nu = 1, 2, \ldots)$, and hence

$$\|\alpha_\nu y_\nu + x_\nu - a\| \geqq \gamma \qquad (\nu = 1, 2, \ldots).$$

Consequently we obtain

$$2|(x_\nu - a, y_\nu)|^2 \leqq \varepsilon_\nu^2\{(\varepsilon_\nu^2 + \gamma^2)\|y_\nu\|^2 + 1\}$$

for every $\nu = 1, 2, \ldots$. This relation yields

$$\lim_{\nu\to\infty}(x_\nu - a, y_\nu) = 0.$$

Therefore we have especially

$$\lim_{\nu\to\infty}(x_\nu - a, x_\nu) = 0,$$
$$\lim_{\nu\to\infty}(x_\nu - a, x) = 0 \qquad \text{for every } x \in A.$$

Since $\lim_{\nu\to\infty}\|x_\nu - a\|^2 = \gamma^2$, we conclude hence

$$\lim_{\nu\to\infty}(x_\nu - a, a) = -\gamma^2.$$

We obtain by the formula §4(3) for every $\nu, \mu = 1, 2, \ldots$

$$\|x_\nu - x_\mu\|^2 = 2(\|x_\nu - a\|^2 + \|x_\mu - a\|^2) - 4\left\|\frac{1}{2}(x_\nu + x_\mu) - a\right\|^2$$
$$\leqq 2(\|x_\nu - a\|^2 + \|x_\mu - a\|^2) - 4\gamma^2$$

because $\frac{1}{2}(x_\nu + x_\mu) \in A$. Thus we conclude

$$\lim_{\nu,\mu\to\infty}\|x_\nu - x_\mu\| = 0,$$

and hence by (2) and (3)

$$\lim_{\nu,\mu\to\infty}\|\varphi_{x_\nu} - \varphi_{x_\mu}\| = 0.$$

Since the adjoint space $\bar{R}$ of R is complete by Theorem 3.1, there exists hence $\bar{x} \in \bar{R}$ such that $\lim_{\nu\to\infty}\varphi_{x_\nu} = \bar{x}$. For such $\bar{x} \in \bar{R}$, putting $\bar{a} = \bar{x} - \varphi_a$, we have by Theorem 3.3 for every $x \in A$

$$\bar{a}(x) = \lim_{\nu\to\infty}(x_\nu, x) - (a, x) = \lim_{\nu\to\infty}(x_\nu - a, x) = 0,$$
$$\bar{a}(a) = \lim_{\nu\to\infty}(x_\nu - a, a) = -\gamma^2 \neq 0.$$

Thus our assertion is established.

For any $\bar{x} \in \bar{R}$, putting

$$A = \{x : \bar{x}(x) = 0\},$$

we see easily that A is a closed linear manifold of R. If there is an element $a \in R$ such that $\bar{x}(a) \neq 0$, then we can find $\bar{a} \in \bar{R}$ and $x_\nu \in R$ $(\nu = 1, 2, \ldots)$ such that

$$\bar{a} = \lim_{\nu\to\infty}(\varphi_{x_\nu} - \varphi_a) = \lim_{\nu\to\infty}\varphi_{x_\nu - a},$$

$$\overline{a}(a) \neq 0, \qquad \overline{a}(x) = 0 \qquad \text{for every } x \in A,$$

as proved just above. Then for every $x \in R$ we have

$$\overline{x}\left(x - \frac{\overline{x}(x)}{\overline{x}(a)}\, a\right) = \overline{x}(x) - \frac{\overline{x}(x)}{\overline{x}(a)}\, \overline{x}(a) = 0,$$

and hence

$$\overline{a}\left(x - \frac{\overline{x}(x)}{\overline{x}(a)}\, a\right) = 0, \text{ that is, } \overline{a}(x) = \frac{\overline{x}(x)}{\overline{x}(a)}\, \overline{a}(a).$$

Consequently, putting $\alpha = \dfrac{\overline{x}(a)}{\overline{a}(a)}$, we have

$$\overline{x} = \alpha\, \overline{a} = \lim_{\nu \to \infty} \varphi_{\alpha(x_\nu - a)}.$$

Therefore we have

Theorem 5.2. *$\{\varphi_x : x \in R\}$ is dense in the adjoint space $\overline{R}$.*

For every Cauchy sequence $x_\nu \in R$ $(\nu = 1, 2, \ldots)$, the number sequence $\|x_\nu\|$ $(\nu = 1, 2, \ldots)$ is convergent, because

$$|\,\|x_\nu\| - \|x_\mu\|\,| \leqq \|x_\nu - x_\mu\| \qquad (\nu, \mu = 1, 2, \ldots).$$

Furthermore, for every two Cauchy sequences x_ν and $y_\nu \in R$ $(\nu = 1, 2, \ldots)$, the number sequence (x_ν, y_ν) $(\nu = 1, 2, \ldots)$ is convergent, because we have by the formula §4(2)

$$|(x_\nu, y_\nu) - (x_\mu, y_\mu)| \leqq \|x_\nu\|\,\|y_\nu - y_\mu\| + \|x_\nu - x_\mu\|\,\|y_\mu\|$$

for every $\nu, \mu = 1, 2, \ldots$.

For every two elements $\overline{x}, \overline{y} \in \overline{R}$, we can find by Theorem 5.2 two sequences x_ν and $y_\nu \in R$ $(\nu = 1, 2, \ldots)$ such that

$$\lim_{\nu \to \infty} \varphi_{x_\nu} = \overline{x}, \qquad \lim_{\nu \to \infty} \varphi_{y_\nu} = \overline{y}.$$

By virtue of the formula (2), we see that such sequences x_ν and $y_\nu \in R$ $(\nu = 1, 2, \ldots)$ are Cauchy sequences. Thus, putting

$$(\overline{x}, \overline{y}) = \lim_{\nu \to \infty} (x_\nu, y_\nu),$$

we see easily that $(\overline{x}, \overline{y})$ $(\overline{x}, \overline{y} \in \overline{R})$ is an inner product on the adjoint space $\overline{R}$, and furthermore we have

$$(4) \qquad (\varphi_x, \varphi_y) = (x, y) \qquad \text{for every } x, y \in R,$$

$$(5) \qquad (\overline{x}, \overline{x}) = \|\overline{x}\|^2 \qquad \text{for every } \overline{x} \in \overline{R},$$

because, if $\lim_{\nu \to \infty} \varphi_{x_\nu} = \overline{x}$, then we have

$$\|\overline{x}\|^2 = \lim_{\nu \to \infty} \|\varphi_{x_\nu}\|^2 = \lim_{\nu \to \infty} \|x_\nu\|^2 = \lim_{\nu \to \infty} (x_\nu, x_\nu) = (\overline{x}, \overline{x}).$$

Thus we have proved that there exists an inner product on the adjoint space $\overline{R}$ subject to the conditions (4) and (5). We can conclude

further by Theorem 4.2 that such an inner product on $\overline{R}$ is determined uniquely. Therefore we define the adjoint space of an inner product space R as an inner product space $\overline{R}$ associated with such an inner product. With this definition we have hence by Theorem 3.1

Theorem 5.3. The adjoint space of an inner product space is a Hilbert space, namely a complete inner product space.

Theorem 5.4 If R is a Hilbert space, then for any bounded anti-linear functional φ on R, we can find $a \in R$ such that

$$\varphi(x) = (a, x) \qquad \text{for every } x \in R.$$

Proof. If R is complete, then we see by the formula (2) that

$$\{ \varphi_x : x \in R \}$$

is a closed manifold of the adjoint space $\overline{R}$ of R. On the other hand $\{ \varphi_x : x \in R \}$ is dense in $\overline{R}$ by Theorem 5.2. Thus we have

$$\overline{R} = \{ \varphi_x : x \in R \}.$$

Therefore, for any $\bar{a} \in \overline{R}$ we can find $a \in R$ such that

$$\bar{a}(x) = (a, x) \qquad \text{for every } x \in R.$$

§6 Completion

Let R and S be two linear space. A one-to-one correspondence T between R and a manifold A of S : $Tx \in A$ $(x \in R)$, is called a transformation of R into S, and A is called the range of T. If the range of T coincides with the whole space S, then T is called a transformation from R to S. A transformation T of R into S is said to be linear, if

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

for every $x, y \in R$ and complex numbers α, β. For two linear spaces R and S, if there is a linear transformation T from R to S, then we say that R is isomorphic to S by T. With this definition, it is evident that the range of a linear transformation is a linear manifold, and for a linear transformation T of R into S, R is isomorphic to the range of T.

For two normed spaces R and S, a transformation T of R into S is said to be isometric, if T is linear and

$$\|Tx\| = \|x\| \qquad \text{for every } x \in R.$$

For two normed spaces R and S, if there is an isometric transformation T from R to S, then R is said to be isometric to S by T.

If both R and S are inner product spaces, then we see by the formula §4(4) that for every isometric transformation T of R into S we have

$$(Tx, Ty) = (x, y) \quad \text{for every } x, y \in R.$$

For an inner product space R, a Hilbert space S is said to be a completion of R by a transformation T of R into S, if T is isometric and the range of T is dense in S. With this definition, for every inner product space R, the adjoint space $\bar{R}$ of R is a completion of R. Because, putting

$$Tx = \varphi_x \qquad (x \in R),$$

we see at once by (2) and (3) in §5 that T is an isometric transformation of R into the adjoint space $\bar{R}$, and the range of T is dense in $\bar{R}$ by Theorem 5.2. Thus we have

Theorem 6.1. The adjoint space $\bar{R}$ of an inner product space R is a completion of R by the transformation

$$Tx = \varphi_x \quad (x \in R), \qquad \varphi_x(y) = (x, y) \quad (x, y \in R).$$

For two completions S_1, S_2 of an inner product space R respectively by transformations T_1 and T_2, putting

$$T_0(T_1 x) = T_2 x \qquad (x \in R),$$

we see easily that T_0 is an isometric transformation from the range of T_1 to that of T_2. Let A_1 and A_2 be ranges of T_1 and T_2 respectively. Since A_1 is dense in S_1, for any $x \in S_1$ we can find $x_\nu \in A_1$ $(\nu = 1, 2, \ldots)$ such that $\lim_{\nu \to \infty} x_\nu = x$. Then $T_0 x_\nu$ $(\nu = 1, 2, \ldots)$ is obviously a Cauchy sequence, and hence we can find uniquely $y \in S_2$ such that $\lim_{\nu \to \infty} T_0 x_\nu = y$. Thus, putting $y = Tx$, we see easily that T is an isometric transformation from S_1 to S_2 and we have $Tx = T_0 x$ for every $x \in A_1$. Therefore we can state

Theorem 6.2. The completion of an inner product space is determined uniquely within an isometric transformation.

§7 Orthonormal Systems

Let R be an inner product space. For two elements $a, b \in R$, we define $a \perp b$ to mean $(a, b) = 0$, and we shall say that a is orthogonal to b, if $a \perp b$. With this definition, we have obviously

(1) $a \perp b$ implies $b \perp a$,

(2) $a \perp b$, $a \perp c$ implies $a \perp b + c$,

(3) $a \perp b$ implies $\alpha a \perp b$ for every complex number α,

(4) $a \perp b$ implies $\|a + b\|^2 = \|a\|^2 + \|b\|^2$,

(5) $a_\nu \perp b$ $(\nu = 1, 2, \ldots)$, $\lim_{\nu \to \infty} a_\nu = a$ implies $a \perp b$.

Zero element 0 is obviously orthogonal to every element. Conversely we have

Theorem 7.1. If a manifold A is fundamental in R and

$$(a, x) = (b, x) \qquad \text{for every } x \in A,$$

then we have $a = b$.

Proof. Denoting by B the linear manifold generated by A, we see by Theorem 2.5 that B is dense in R, and it is obvious by assumption that $(a, x) = (b, x)$ for every $x \in B$. Thus we conclude by Theorem 4.2 that $(a, x) = (b, x)$ for every $x \in R$. Therefore we have

$$\|a - b\|^2 = (a, a - b) - (b, a - b) = 0,$$

and hence $a = b$.

For a manifold A of R, we define $a \perp A$ to mean that

$$a \perp x \qquad \text{for every } x \in A,$$

and we shall say that an element a is orthogonal to A, if $a \perp A$.

Theorem 7.2. If a linear manifold A of R is complete, then for any $a \overline{\in} A$ we can find $x \in A$ and a complex number α such that

$$\alpha a + x \perp A, \qquad \|\alpha a + x\| = 1.$$

Proof. Denoting by B the linear manifold generated by $\{a, A\}$, we see by Theorem 2.8 that B is complete. Therefore we can consider B itself as a Hilbert space and A as a closed linear manifold of B. Then we conclude by Theorems 5.1 and 5.4 that there exists an element

$\ell \in B$ such that $\ell \perp A$ and $(\ell, a) \neq 0$. For such $\ell \in B$, since $\ell \neq 0$, putting $c = \frac{1}{\|\ell\|}\ell$, we have $c \in B$, $c \perp A$ and $\|c\| = 1$. Furthermore, since B is the linear manifold generated by $\{a, A\}$, we can find $x \in A$ and a complex number α such that $c = x + \alpha a$.

A system $a_\lambda \in R$ $(\lambda \in \Lambda)$ is said to be an orthogonal system, if $a_\lambda \perp a_\rho$ for $\lambda \neq \rho$. An orthogonal system $a_\lambda \in R$ $(\lambda \in \Lambda)$ is said to be an orthonormal system, if $\|a_\lambda\| = 1$ for every $\lambda \in \Lambda$.

Every orthonormal system $c_\lambda \in R$ $(\lambda \in \Lambda)$ is linearly independent. Because, $\sum_{\nu=1}^{\kappa} \alpha_\nu c_{\lambda_\nu} = 0$ implies

$$\alpha_\mu = \sum_{\nu=1}^{\kappa} \alpha_\nu (c_{\lambda_\nu}, c_{\lambda_\mu}) = \left(\sum_{\nu=1}^{\kappa} \alpha_\nu c_{\lambda_\nu}, c_{\lambda_\mu}\right) = 0$$

for every $\mu = 1, 2, \ldots, \kappa$. Recalling Theorem 2.9, we obtain immediately by Theorem 7.2

Theorem 7.3. For a linearly independent sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$, we can find an orthonormal system $c_\nu \in R$ $(\nu = 1, 2, \ldots)$ such that

$$c_\nu = \sum_{\mu=1}^{\nu} \alpha_{\nu,\mu} a_\mu, \qquad \alpha_{\nu,\nu} \neq 0 \quad (\nu = 1, 2, \ldots).$$

Recalling Theorem 2.5, we conclude by the formulas (2), (3), (5)

Theorem 7.4. If an element a is orthogonal to a manifold A, then a also is orthogonal to the closed linear manifold generated by A.

Theorem 7.5. For an orthonormal system $c_\lambda \in R$ $(\lambda \in \Lambda)$ we have

$$\|x\|^2 \geqq \sum_{\lambda \in \Lambda} |(x, c_\lambda)|^2 \quad \text{for every } x \in R.$$

Proof. For every finite number of different elements $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots, \kappa)$, putting $\alpha_\lambda = (x, c_\lambda)$ $(\lambda \in \Lambda)$, we have

$$\begin{aligned} \|x - \sum_{\nu=1}^{\kappa} \alpha_{\lambda_\nu} c_{\lambda_\nu}\|^2 &= \|x\|^2 - 2\Re \sum_{\nu=1}^{\kappa} (x, \alpha_{\lambda_\nu} c_{\lambda_\nu}) \\ &\quad + \sum_{\nu,\mu=1}^{\kappa} (\alpha_{\lambda_\nu} c_{\lambda_\nu}, \alpha_{\lambda_\mu} c_{\lambda_\mu}) \\ &= \|x\|^2 - \sum_{\nu=1}^{\kappa} |\alpha_{\lambda_\nu}|^2. \end{aligned}$$

Thus we obtain $\|x\|^2 \geqq \sum_{\nu=1}^{\kappa} |\alpha_{\lambda_\nu}|^2$ for every finite number of different elements $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots,)$, and hence $\|x\|^2 \geqq \sum_{\lambda \in \Lambda} |\alpha_\lambda|^2$.

From this Theorem 7.5 we conclude immediately.

Theorem 7.6. For every orthonormal system $c_\lambda \in R$ $(\lambda \in \Lambda)$

and an arbitrary element $x \in R$, we have $(x, c_\lambda) = 0$ up to at most countable elements of Λ.

For an orthogonal sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$, if $\sum_{\nu=1}^{\infty} a_\nu$ is convergent, then for any permutation $\rho_1, \rho_2, \ldots$ of $1, 2, \ldots$, $\sum_{\nu=1}^{\infty} a_{\rho_\nu}$ also is convergent and

$$\sum_{\nu=1}^{\infty} a_\nu = \sum_{\nu=1}^{\infty} a_{\rho_\nu}.$$

Because, putting $a = \sum_{\nu=1}^{\infty} a_\nu$, we have by the formula (4) and Theorem 2.2

$$\|a\|^2 = \sum_{\nu=1}^{\infty} \|a_\nu\|^2.$$

On the other hand, for any $\varkappa = 1, 2, \ldots$ we can find σ such that $\rho_1, \rho_2, \ldots, \rho_\sigma$ contains $1, 2, \ldots, \varkappa$. Then we have by (4)

$$\|a - \sum_{\nu=1}^{\sigma} a_{\rho_\nu}\|^2 \leqq \sum_{\nu=\varkappa+1}^{\infty} \|a_\nu\|^2,$$

and hence $a = \sum_{\nu=1}^{\infty} a_{\rho_\nu}$ by definition.

Therefore, for an orthogonal system $a_\lambda \in R$ $(\lambda \in \Lambda)$, we define $a = \sum_{\lambda \in \Lambda} a_\lambda$ to mean that there exists a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ such that $a_\lambda = 0$ for $\lambda \neq \lambda_\nu$ $(\nu = 1, 2, \ldots)$ and $a = \sum_{\nu=1}^{\infty} a_{\lambda_\nu}$, and then we shall say that $\sum_{\lambda \in \Lambda} a_\lambda$ is convergent.

With this definition we have obviously by Theorem 4.2

Theorem 7.7. For an orthonormal system $c_\lambda \in R$ $(\lambda \in \Lambda)$, if $a = \sum_{\lambda \in \Lambda} \alpha_\lambda c_\lambda$, then we have

$$\|a\|^2 = \sum_{\lambda \in \Lambda} |\alpha_\lambda|^2, \qquad (a, c_\lambda) = \alpha_\lambda \text{ for every } \lambda \in \Lambda.$$

Furthermore we conclude by Theorems 2.1 and 4.2

Theorem 7.8. For an orthonormal system $c_\lambda \in R$ $(\lambda \in \Lambda)$, if $a = \sum_{\lambda \in \Lambda} \alpha_\lambda c_\lambda$, $b = \sum_{\lambda \in \Lambda} \beta_\lambda c_\lambda$, then we have

$$(a, b) = \sum_{\lambda \in \Lambda} \alpha_\lambda \beta_\lambda, \qquad \sum_{\lambda \in \Lambda} |\alpha_\lambda \beta_\lambda| < +\infty,$$

$$\alpha a + \beta b = \sum_{\lambda \in \Lambda} (\alpha \alpha_\lambda + \beta \beta_\lambda) c_\lambda, \text{ for complex numbers } \alpha, \beta.$$

Because $\sum_{\lambda \in \Lambda} |\alpha_\lambda \beta_\lambda| \leqq \left(\sum_{\lambda \in \Lambda} |\alpha_\lambda|^2 \sum_{\lambda \in \Lambda} |\beta_\lambda|^2 \right)^{\frac{1}{2}} < +\infty.$

If $a = \sum_{\lambda \in \Lambda} \alpha_\lambda c_\lambda$, then a is contained obviously in the closed linear manifold generated by $c_\lambda \in R$ $(\lambda \in \Lambda)$. Conversely we have

Theorem 7.9. For each element a of the closed linear manifold generated by an orthonormal system $c_\lambda \in R$ $(\lambda \in \Lambda)$ we can find uniquely complex numbers α_λ $(\lambda \in \Lambda)$ such that $a = \sum_{\lambda \in \Lambda} \alpha_\lambda c_\lambda$.

Proof. If we put $\alpha_\lambda = (a, c_\lambda)$ $(\lambda \in \Lambda)$, then for every finite number of complex numbers ξ_ν and different elements $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots \kappa)$ we have

$$\| a - \sum_{\nu=1}^{\kappa} \xi_\nu c_{\lambda_\nu} \|^2 = \| a \|^2 - \sum_{\nu=1}^{\kappa} \bar{\xi}_\nu \alpha_{\lambda_\nu} - \sum_{\nu=1}^{\kappa} \xi_\nu \overline{\alpha_{\lambda_\nu}} + \sum_{\nu=1}^{\kappa} \xi_\nu \bar{\xi}_\nu$$
$$= \| a \|^2 - \sum_{\nu=1}^{\kappa} |\alpha_{\lambda_\nu}|^2 + \sum_{\nu=1}^{\kappa} |\alpha_{\lambda_\nu} - \xi_\nu|^2 \geqq \| a \|^2 - \sum_{\nu=1}^{\kappa} |\alpha_{\lambda_\nu}|^2.$$

Since a is contained in the closed linear manifold generated by $c_\lambda \in R$ $(\lambda \in \Lambda)$, we conclude hence by Theorem 2.5

$$\| a \|^2 = \sum_{\lambda \in \Lambda} |\alpha_\lambda|^2.$$

Thus we can find a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ such that $\alpha_\lambda = 0$ for $\lambda \neq \lambda_\nu$ $(\nu = 1, 2, \ldots)$, and then we have

$$\| a - \sum_{\nu=1}^{\kappa} \alpha_{\lambda_\nu} c_{\lambda_\nu} \|^2 = \| a \|^2 - \sum_{\nu=1}^{\kappa} |\alpha_{\lambda_\nu}|^2.$$

Therefore we conclude $a = \sum_{\nu=1}^{\infty} \alpha_{\lambda_\nu} c_{\lambda_\nu}$, and hence $a = \sum_{\lambda \in \Lambda} \alpha_\lambda c_\lambda$. The uniqueness of such α_λ $(\lambda \in \Lambda)$ is evident by Theorem 7.7.

Theorem 7.10. In order that the closed linear manifold generated by an orthonormal system $c_\lambda \in R$ $(\lambda \in \Lambda)$ be complete, it is necessary and sufficient that $\sum_{\lambda \in \Lambda} |\alpha_\lambda|^2 < +\infty$ implies convergence of $\sum_{\lambda \in \Lambda} \alpha_\lambda c_\lambda$.

Proof. For $\sum_{\lambda \in \Lambda} |\alpha_\lambda|^2 < +\infty$, we can find a sequence $\lambda_\nu \in \Lambda$ such that $\alpha_\lambda = 0$ for $\lambda \neq \lambda_\nu$ $(\nu = 1, 2, \ldots)$, and

$$\| \sum_{\nu=\mu}^{\sigma} \alpha_{\lambda_\nu} c_{\lambda_\nu} \|^2 = \sum_{\nu=\mu}^{\sigma} |\alpha_{\lambda_\nu}|^2 \quad \text{for } \sigma > \mu.$$

Thus, if the closed linear manifold generated by c_λ $(\lambda \in \Lambda)$ is complete, then $\sum_{\nu=1}^{\infty} \alpha_{\lambda_\nu} c_{\lambda_\nu}$ is convergent, and hence $\sum_{\lambda \in \Lambda} \alpha_\lambda c_\lambda$ is convergent.

Conversely, let $a_\nu = \sum_{\lambda \in \Lambda} \alpha_{\nu,\lambda} c_\lambda$ $(\nu = 1, 2, \ldots)$ be a Cauchy sequence. Since we have by Theorems 7.7 and 7.8

$$\| a_\nu - a_\mu \|^2 = \sum_{\lambda \in \Lambda} |\alpha_{\nu,\lambda} - \alpha_{\mu,\lambda}|^2,$$

we conclude that $\alpha_{\nu,\lambda}$ $(\nu = 1, 2,\ldots)$ is convergent for every $\lambda \in \Lambda$. Furthermore, putting $\alpha_\lambda = \lim_{\nu\to\infty} \alpha_{\nu,\lambda}$, we see easily by Theorem 7.7

$$\lim_{\nu\to\infty} \| a_\nu \|^2 = \lim_{\nu\to\infty} \sum_{\lambda\in\Lambda} |\alpha_{\nu,\lambda}|^2 \geqq \sum_{\lambda\in\Lambda} |\alpha_\lambda|^2.$$

Therefore, if the condition stated in Theorem 7.10 is satisfied, then we can put $a = \sum_{\lambda\in\Lambda} \alpha_\lambda c_\lambda$, and we have by Theorems 7.7 and 7.8

$$\| a_\nu - a \|^2 = \sum_{\lambda\in\Lambda} |\alpha_{\nu,\lambda} - \alpha_\lambda|^2 \qquad (\nu = 1, 2,\ldots).$$

On the other hand we have

$$\lim_{\mu\to\infty} \| a_\nu - a_\mu \|^2 \geqq \sum_{\lambda\in\Lambda} |\alpha_{\nu,\lambda} - \alpha_\lambda|^2 \quad (\nu = 1, 2,\ldots).$$

Thus we conclude $\lim_{\nu\to\infty} \| a_\nu - a \| = 0$, because a_ν $(\nu = 1, 2,\ldots)$ is a Cauchy sequence by assumption. Therefore the closed linear manifold generated by c_λ $(\lambda \in \Lambda)$ is complete.

Recalling Theorem 2.7, we conclude immediately from Theorem 7.10

Theorem 7.11. *In a Hilbert space* R, *for any orthogonal system* $a_\lambda \in R$ $(\lambda \in \Lambda)$, $\sum_{\lambda\in\Lambda} a_\lambda$ *is convergent, if and only if*

$$\sum_{\lambda\in\Lambda} \| a_\lambda \|^2 < +\infty.$$

§8 Dimensions of Hilbert Spaces

Let R be a Hilbert space. An orthonormal system $c_\lambda \in R$ $(\lambda \in \Lambda)$ is said to be *complete*, if $(x, c_\lambda) = 0$ for all $\lambda \in \Lambda$ implies $x = 0$. We see easily by Maximal Theorem that there exists a complete orthonormal system for every Hilbert space.

Theorem 8.1. *In order that an orthonormal system* c_λ $(\lambda \in \Lambda)$ *be complete, it is necessary and sufficient that* c_λ $(\lambda \in \Lambda)$ *is fundamental in* R.

Proof. Let S be the closed linear space generated by c_λ $(\lambda \in \Lambda)$. Since S is complete by Theorem 2.7, if S does not coincide with the whole space R, then we can find by Theorem 7.2 an element $a \in R$ such that $\| a \| = 1$, $(a, c_\lambda) = 0$ for all $\lambda \in \Lambda$, and hence c_λ $(\lambda \in \Lambda)$ is not complete by definition. If $S = R$, then $(a, c_\lambda) = 0$ for all $\lambda \in \Lambda$ implies $a = 0$ by Theorem 7.1.

By virtue of Theorems 8.1 and 7.9, we obtain immediately

Theorem 8.2. *For a complete orthonormal system* c_λ $(\lambda \in \Lambda)$

of a Hilbert space R, every $a \in R$ may be represented uniquely in the form $a = \sum_{\lambda \in \Lambda} \alpha_\lambda c_\lambda$.

Theorem 8.3. For a Hilbert space R, all complete orthonormal systems in R have the same density.

Proof. Firstly we consider the case where R has a complete orthonormal system compsed only of a finite number of elements. Let $c_\nu \in R\ (\nu = 1, 2, \ldots, \varkappa)$ be a complete orthonormal system in R. We shall prove by induction that every $\varkappa + 1$ elements are not linearly independent. In fact, it is evident in the case: $\varkappa = 1$. We assume that our assertion is valid for $\varkappa - 1$. For a system $a_\mu \in R$ $(\mu = 1, 2, \ldots, \varkappa + 1)$ we can find by Theorem 7.9 $\alpha_{\mu,\nu}$ such that

$$a_\mu = \sum_{\nu=1}^{\varkappa} \alpha_{\mu,\nu} c_\nu \qquad (\mu = 1, 2, \ldots, \varkappa + 1).$$

We can suppose obviously $\alpha_{\varkappa+1,\varkappa} \neq 0$, without losing the generality. Then we have

$$a_\mu - \frac{\alpha_{\mu,\varkappa}}{\alpha_{\varkappa+1,\varkappa}} a_{\varkappa+1} = \sum_{\nu=1}^{\varkappa-1} \alpha_{\mu,\nu} c_\nu \ (\mu = 1, 2, \ldots, \varkappa).$$

Thus we can find by assumption $\gamma_\mu\ (\mu = 1, 2, \ldots, \varkappa)$ such that

$$\sum_{\mu=1}^{\varkappa} |\gamma_\mu| \neq 0, \text{ and } \sum_{\mu=1}^{\varkappa} \gamma_\mu \left(a_\mu - \frac{\alpha_{\mu,\varkappa}}{\alpha_{\varkappa+1,\varkappa}} a_{\varkappa+1} \right) = 0,$$

that is,

$$\sum_{\mu=1}^{\varkappa} \gamma_\mu a_\mu - \left(\sum_{\mu=1}^{\varkappa} \gamma_\mu \frac{\alpha_{\mu,\varkappa}}{\alpha_{\varkappa+1,\varkappa}} \right) a_{\varkappa+1} = 0.$$

Therefore $a_\mu (\mu = 1, 2, \ldots, \varkappa + 1)$ is not linearly independent.

Since every orthonormal system is linearly independent, we conclude hence that every complete orthonormal system is composed of $\varkappa$ elements.

Next we consider the case where R has a complete orthonormal system composed of infinite elements. Then every complete orthonormal system is composed of infinite elements, as proved just above. Let a_λ $(\lambda \in \Lambda)$ and b_γ $(\gamma \in \Gamma)$ be two complete orthonormal system in R. For each $\lambda \in \Lambda$ we can find by Theorem 7.6 at most countable elements $\gamma_{\lambda,\nu} \in \Gamma$ $(\nu = 1, 2, \ldots)$ such that $(b_\gamma, a_\lambda) = 0$ for $\gamma \neq \gamma_{\lambda,\nu}$ $(\nu = 1, \ldots)$. Furthermore, for any $\gamma \in \Gamma$ we can find by definition $\lambda \in \Lambda$ such that $(b_\gamma, a_\lambda) \neq 0$. Therefore we have

$$\sum_{\lambda \in \Lambda} \{ \gamma_{\lambda,1}, \gamma_{\lambda,2}, \ldots \} = \Gamma.$$

Thus, denoting by m and n cardinal numbers of Λ and Γ respectively,

we have hence $m \leqq \aleph_0 m = m$. We also can prove likewise $m \leqq n$. Consequently we have $m = n$.

For a Hilbert space R, all complete orthonormal systems in R have the same density, as proved just above. This same density is called the dimension of R.

Theorem 8.4. A Hilbert space R is isometric to a Hilbert space S, if and only if the dimension of R coincides with that of S.

Proof. If R is isometric to S by a transformation $Tx \in S$ $(x \in R)$, then for a complete orthonormal system $e_\lambda \in R$ $(\lambda \in \Lambda)$, we also obtain a complete orthonormal system $Te_\lambda \in S$ $(\lambda \in \Lambda)$. Therefor the dimension of R coincides with that of S.

Conversely, we assume that the dimension of R coincides with that of S, and let a_λ and b_λ $(\lambda \in \Lambda)$ be complete orthonormal systems in R and S respectively. Corresponding to every $x \in R$, we can find by Theorem 7.9 uniquely complex numbers α_λ $(\lambda \in \Lambda)$ such that $x = \sum_{\lambda \in \Lambda} \alpha_\lambda a_\lambda$, and then we obtain by Theorem 7.10 uniquely $y \in S$ such that $y = \sum_{\lambda \in \Lambda} \alpha_\lambda b_\lambda$. Therefore, putting

$$\sum_{\lambda \in \Lambda} \alpha_\lambda b_\lambda = T \sum_{\lambda \in \Lambda} \alpha_\lambda a_\lambda \qquad \text{for } \sum_{\lambda \in \Lambda} |\alpha_\lambda|^2 < +\infty,$$

we see easily by Theorem 7.7 that T is an isometric transformation from R to S, and hence R is isometric to S by T.

A Hilbert space R is said to be separable, if we can find a sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ which is dense in R. With this definition, we have

Theorem 8.5. In order that a Hilbert space R be separable, it is necessary and sufficient that the dimension of R is at most countable.

Proof. If R is separable and a sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ is dense in R, then a_ν $(\nu = 1, 2, \ldots)$ is obviously fundamental in R. By virtue of Theorem 7.3, we can find an orthonormal system composed of at most countable elements $e_\nu \in R$ $(\nu = 1, 2, \ldots)$ such that every a_ν is a linear combination from e_ν $(\nu = 1, 2, \ldots)$. Then e_ν $(\nu = 1, 2, \ldots)$ is obviously fundamental in R, and hence a complete orthonormal system in R.

Conversely, if we can find a system of at most countable elements $c_\nu \in R$ $(\nu = 1, 2, \ldots)$ which is fundamental in R, then, denoting by A the totality of linear combinations with coefficients of rational complex numbers from c_ν $(\nu = 1, 2, \ldots)$, we see easily that A is countable and dense in R. Since every complete orthonormal system in R is obviously fundamental in R, we obtain therefore our assertion. Furthermore we also have proved

Theorem 8.6. *If a Hilbert space R has a fundamental system composed of at most countable elements, then R is separable and the dimension of R is at most countable.*

§9 Weak Convergence

Let R be an inner product space. Corresponding to every $a \in R$, putting $\varphi_a(x) = (a, x)$ $(x \in R)$, we obtain a bounded anti-linear functional on R, as stated in §5. A sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ is said to be *weakly convergent* to a *limit* $a \in R$, and we write

$$\text{w-}\lim_{\nu\to\infty} a_\nu = a,$$

if φ_{a_ν} $(\nu = 1, 2, \ldots)$ is weakly convergent to φ_a, as defined in §3, that is, if $\lim_{\nu\to\infty} (a_\nu, x) = (a, x)$ for every $x \in R$.

With this definition, we see at once by Theorem 7.1 that the limit of a weakly convergent sequence is determined uniquely. Furthermore we have obviously

Theorem 9.1. $\text{w-}\lim_{\nu\to\infty} a_\nu = a$, $\text{w-}\lim_{\nu\to\infty} b_\nu = b$ *implies*

$$\text{w-}\lim_{\nu\to\infty} (\alpha a_\nu + \beta b_\nu) = \alpha a + \beta b$$

for every complex numbers α, β. $\text{w-}\lim_{\nu\to\infty} a_\nu = a$, $\lim_{\nu\to\infty} \alpha_\nu = \alpha$ *implies* $\text{w-}\lim_{\nu\to\infty} \alpha_\nu a_\nu = \alpha a$.

Since $\|\varphi_a\| = \|a\|$ by §5(2), we obtain by Theorem 3.4

Theorem 9.2. $\text{w-}\lim_{\nu\to\infty} a_\nu = a$ *implies* $\varliminf_{\nu\to\infty} \|a_\nu\| \geqq \|a\|$.

We have obviously by Theorem 3.3 that $\lim_{\nu\to\infty} a_\nu = a$ implies $\text{w-}\lim_{\nu\to\infty} a_\nu = a$. Therefore usual convergence may be often called *strong convergence* against weak convergence.

Theorem 9.3. *For a fundamental manifold A of R, if*

$$\lim_{\nu\to\infty}(a_\nu, x) = (a, x) \qquad \text{for every } x \in A,$$

$$\sup_{\nu\geq 1} \|a_\nu\| < +\infty,$$

then we have $\text{w-}\lim_{\nu\to\infty} a_\nu = a$.

Proof. By virtue of Theorem 3.6, we conclude from our assumption that there exists a bounded anti-linear functional φ on R such that $\text{w-}\lim_{\nu\to\infty} \varphi_{a_\nu} = \varphi$. For such φ, we have by assumption that $\varphi(x) = \varphi_a(x)$ for every $x \in A$. Since A is fundamental in R and both φ and φ_a are continuous, we conclude hence that $\varphi(x) = \varphi_a(x)$ for every $x \in R$, that is, $\varphi = \varphi_a$. Therefore we have by definition $\text{w-}\lim_{\nu\to\infty} a_\nu = a$.

Theorem 9.4. $\text{w-}\lim_{\nu\to\infty} a_\nu = a$, $\sup_{\nu\geq 1}\|a_\nu\| < +\infty$, $\lim_{\nu\to\infty} b_\nu = b$ implies $\lim_{\nu\to\infty}(a_\nu, b_\nu) = (a, b)$.

Proof. Since $|(a_\nu, b_\nu) - (a, b)| \leq |(a_\nu, b_\nu - b)| + |(a_\nu, b) - (a, b)| \leq \|a_\nu\|\|b_\nu - b\| + |(a_\nu, b) - (a, b)|$, we obtain by assumption $\lim_{\nu\to\infty}(a_\nu, b_\nu) = (a, b)$.

Theorem 9.5. If $\text{w-}\lim_{\nu\to\infty} a_\nu = a$, $\lim_{\nu\to\infty}\|a_\nu\| = \|a\|$, then we have $\lim_{\nu\to\infty} a_\nu = a$.

Proof. $\text{w-}\lim_{\nu\to\infty} a_\nu = a$ implies $\text{w-}\lim_{\nu\to\infty}(a_\nu + a) = 2a$ and hence $2\|a\| \leq \varliminf_{\nu\to\infty}\|a_\nu + a\|$ by Theorem 9.2. Since we have by the formula §4(3)

$$\|a_\nu - a\|^2 = 2(\|a\|^2 + \|a_\nu\|^2) - \|a_\nu + a\|^2$$

and $\lim_{\nu\to\infty}\|a_\nu\| = \|a\|$ by assumption, we conclude

$$\overline{\lim_{\nu\to\infty}}\|a_\nu - a\|^2 = 4\|a\|^2 - \varliminf_{\nu\to\infty}\|a_\nu + a\|^2 \leq 0.$$

Therefore we have $\lim_{\nu\to\infty} a_\nu = a$.

We shall consider in the sequel the case where R is a Hilbert space. Recalling Theorem 3.2, we have obviously

Theorem 9.6. In a Hilbert space R, if $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ is weakly convergent, then $\sup_{\nu\geq 1}\|a_\nu\| < +\infty$.

Therefore we conclude by Theorem 9.4

Theorem 9.7. In a Hilbert space R, $\text{w-}\lim_{\nu\to\infty} a_\nu = a$, $\lim_{\nu\to\infty} b_\nu = b$ implies $\lim_{\nu\to\infty}(a_\nu, b_\nu) = (a, b)$.

Theorem 9.8. For a fundamental manifold A of a Hilbert space R, if (a_ν, x) $(\nu = 1, 2, \ldots)$ is convergent for every $x \in A$ and $\sup_{\nu \geq 1} \|a_\nu\| < +\infty$, then a_ν $(\nu = 1, 2, \ldots)$ is weakly convergent.

Proof. Recalling Theorem 3.6, we can conclude from assumption that there exists a bounded anti-linear functional φ on R such that

$$\lim_{\nu \to \infty} (a_\nu, x) = \varphi(x) \qquad \text{for every } x \in R.$$

Since R is a Hilbert space by assumption, we can find then by Theorem 5.4 $a \in R$ such that $(a, x) = \varphi(x)$ for every $x \in R$. Therefore a_ν $(\nu = 1, 2, \ldots)$ is weakly convergent by definition.

Recalling Theorem 3.2, we obtain by Theorem 9.8

Theorem 9.9. In a Hilbert space R, if (a_ν, x) $(\nu = 1, 2, \ldots)$ is convergent for every $x \in R$, then a_ν $(\nu = 1, 2, \ldots)$ is weakly convergent.

Theorem 9.10. In a Hilbert space R, if (a_ν, a_μ) $(\nu = 1, 2, \ldots)$ is convergent for every $\mu = 1, 2, \ldots$ and $\sup_{\nu \geq 1} \|a_\nu\| < +\infty$, then a_ν $(\nu = 1, 2, \ldots)$ is weakly convergent.

Proof. Denoting by B the closed linear manifold generated by a_ν $(\nu = 1, 2, \ldots)$, we see by Theorem 2.7 that B is complete, because R is complete by assumption. Therefore, putting $A = \{x : x \perp B\}$, we see by Theorem 7.2 that the union $A + B$ is fundamental, and hence $\{A, a_1, a_2, \ldots\}$ also is fundamental. Since (a_ν, a_μ) $(\nu = 1, 2, \ldots)$ is convergent for every $\mu = 1, 2, \ldots$, and $(a_\nu, x) = 0$ for every $x \in A$ and $\nu = 1, 2, \ldots$, we conclude by Theorem 9.8 that a_ν $(\nu = 1, 2, \ldots)$ is weakly convergent.

Theorem 9.11. In a Hilbert space R, if $\sup_{\nu \geq 1} \|a_\nu\| < +\infty$, then a_ν $(\nu = 1, 2, \ldots)$ contains a weakly convergent subsequence.

Proof. Putting $\gamma = \sup_{\nu \geq 1} \|a_\nu\|$, we have by §4(2)

$$|(a_\nu, a_\mu)| \leqq \|a_\nu\| \|a_\mu\| \leqq \gamma^2 \qquad \text{for every } \nu, \mu = 1, 2, \ldots,$$

and hence we can find a subsequence a_{ν_ρ} $(\rho = 1, 2, \ldots)$ such that (a_{ν_ρ}, a_μ) $(\rho = 1, 2, \ldots)$ is convergent for every $\mu = 1, 2, \ldots$. Then a_{ν_ρ} $(\rho = 1, 2, \ldots)$ is weakly convergent by Theorem 9.10.

Recalling Theorem 3.2, we conclude easily

Theorem 9.12. In a Hilbert space R, for a system $a_\lambda \in R$ $(\lambda \in \Lambda)$, if $\sup_{\lambda \in \Lambda} |(a_\lambda, x)| < +\infty$ for every $x \in R$, then we have

$$\sup_{\lambda \in \Lambda} \| a_\lambda \| < +\infty .$$

Theorem 9.13. If $\text{w-}\lim_{\nu \to \infty} a_\nu = a$, and all a_ν $(\nu = 1, 2, \ldots)$ are contained in a linear manifold M generated by a finite number of elements, then $\lim_{\nu \to \infty} a_\nu = a$.

Proof. By assumption, we can find an orthonormal system $c_\mu \in M$ $(\mu = 1, 2, \ldots, \varkappa)$ such that every $x \in M$ may be repsented uniquely in the form $x = \sum_{\mu=1}^{\varkappa} \alpha_\mu c_\mu$. Putting $a_\nu = \sum_{\mu=1}^{\varkappa} \alpha_{\nu,\mu} c_\mu$ $(\nu = 1, 2, \ldots)$, we obtain by assumption for every $\mu = 1, 2, \ldots,$

$$\lim_{\nu \to \infty} \alpha_{\nu,\mu} = \lim_{\nu \to \infty} (a_\nu, c_\mu) = (a, c_\mu)$$

and hence

$$\lim_{\nu \to \infty} a_\nu = \lim_{\nu \to \infty} \sum_{\mu=1}^{\varkappa} \alpha_{\nu,\mu} c_\mu = \sum_{\mu=1}^{\varkappa} (a, c_\mu) c_\mu .$$

Thus we have $\lim_{\nu \to \infty} a_\nu = a$.

§10 Operators

Let R be a linear space. For a manifold D of R, a mapping T of D into R: $Tx \in R$ $(x \in D)$, is said to be an operator in R. Here D is called the domain of T, and $\{Tx : x \in D\}$ the range of T.

An operator T in R with domain D is said to be linear, if D is a linear manifold of R and

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

for every $x, y \in D$ and complex numbers α, β. With this definition, it is obvious that the range of an linear operator T in R is a linear manifold of R and $T0 = 0$.

Now let R be a normed space. An operator T in R with domain D is said to be continuous, if $\lim_{\nu \to \infty} a_\nu = a$, $a, a_\nu \in D$ $(\nu = 1, 2, \ldots)$, implies $\lim_{\nu \to \infty} Ta_\nu = Ta$. An operator T in R with domain D is said to be bounded, if $\sup_{\|x\| \leqq \alpha, x \in D} \| Tx \| < +\infty$ for every positive number α. With this definition, we see easily that a linear operator T with domain D is bounded, if and only if

$$\sup_{\|x\| \leqq 1, x \in D} \| Tx \| < +\infty ,$$

Thus for a bounded linear operator T with domain D we define the norm

$\|T\|$ to mean

(1) $$\|T\| = \sup_{\|x\| \leq 1,\ x \in D} \|Tx\|.$$

With this definition we have obviously

(2) $$\|Tx\| \leq \|T\| \|x\| \qquad \text{for every } x \in D.$$

Theorem 10.1. *In order that a linear operator T be continuous, it is necessary and sufficient that T is bounded.*

Proof. If a linear operator T with domain D is not bounded, then we can find by definition (1) $a_\nu \in D$ $(\nu = 1, 2, \ldots)$ such that

$$\|a_\nu\| \leq 1, \qquad \|Ta_\nu\| \geq \nu \qquad (\nu = 1, 2, \ldots).$$

Then we have $\frac{1}{\nu} a_\nu \in D$, $\lim_{\nu \to \infty} \frac{1}{\nu} a_\nu = 0$, but $\|T(\frac{1}{\nu} a_\nu)\| \geq 1$ for $\nu = 1, 2, \ldots$. Therefore, if T is continuous, then T is bounded.

Conversely, if T is bounded, then $\lim_{\nu \to \infty} a_\nu = a$, a, $a_\nu \in D$ $(\nu = 1, 2, \ldots)$ implies by the formula (2)

$$\lim_{\nu \to \infty} \|Ta_\nu - Ta\| = \lim_{\nu \to \infty} \|T(a_\nu - a)\| \leq \lim_{\nu \to \infty} \|T\| \|a_\nu - a\| = 0,$$

and hence $\lim_{\nu \to \infty} Ta_\nu = Ta$. Therefore T is continuous by definition.

If the domain of an operator T coincides with the whole space R, then we shall say that T is an operator *on* R. For two operators T and S on R, we define an operator TS to mean

$$TSx = T(Sx) \qquad (x \in R);$$

$T + S$ to mean

$$(T + S)x = Tx + Sx \qquad (x \in R);$$

and αT for every complex number α to mean

$$(\alpha T)x = \alpha(Tx) \qquad (x \in R).$$

If $TS = ST$, then we shall say that T is *commutative* to S.

Every complex number ξ may be considered itself as an operator on R: $\xi x \in R$ $(x \in R)$. Then ξ is obviously a bounded linear operator on R and $\|\xi\| = |\xi|$.

For two bounded linear operators T and S on R, we see easily by definition that all TS, $T + S$, and αT also are bounded linear operators on R for every complex number α and we have

(3) $$\|TS\| \leq \|T\| \|S\|,$$

(4) $$\|T + S\| \leq \|T\| + \|S\|,$$

(5) $$\|\alpha T\| = |\alpha| \|T\|.$$

Thus the totality of bounded linear operators on R constitutes a normed space. Therefore we can consider convergence of sequences of bounded linear operators on R. A sequence of bounded linear operators T_ν $(\nu = 1, 2, \ldots)$ on R is said to be uniformly convergent to a bounded linear operator T on R, if

$$\lim_{\nu\to\infty} \|T_\nu - T\| = 0.$$

With this definition we have obviously by the formula (3)

Theorem 10.2. *If two sequences of bounded linear operators T_ν and S_ν $(\nu = 1, 2, \ldots)$ on R are uniformly convergent to bounded linear operators T and S on R respectively, then $T_\nu S_\nu$ $(\nu = 1, 2, \ldots)$ also is uniformly convergent to TS.*

For a sequence of operators T_ν $(\nu = 1, 2, \ldots)$ on R, if $T_\nu x$ $(\nu = 1, 2, \ldots)$ is convergent for every $x \in R$, then, putting

$$Tx = \lim_{\nu\to\infty} T_\nu x \qquad (x \in R),$$

we obtain an operator T on R, and then we shall say that T_ν $(\nu = 1, 2, \ldots)$ is convergent to T, denoting

$$T = \lim_{\nu\to\infty} T_\nu .$$

Here, if all T_ν $(\nu = 1, 2, \ldots)$ are linear, then its limit T also is obviously linear. Furthermore we have obviously by Theorem 2.2

Theorem 10.3. *For a sequence of bounded linear operators T_ν $(\nu = 1, 2, \ldots)$ on R, if $T = \lim_{\nu\to\infty} T_\nu$ and $\sup_{\nu\geq 1} \|T_\nu\| < +\infty$, then T also is a bounded linear operator on R and*

$$\|T\| \leq \lim_{\nu\to\infty} \|T_\nu\| .$$

Theorem 10.4. *For two sequence of bounded linear operators T_ν and S_ν $(\nu = 1, 2, \ldots)$ on R, if*

$$\lim_{\nu\to\infty} T_\nu = T, \quad \lim_{\nu\to\infty} S_\nu = S, \quad \sup_{\nu\geq 1} \|T_\nu\| < +\infty,$$

then we have $\lim_{\nu\to\infty} T_\nu S_\nu = TS$.

Proof. Putting $\gamma = \sup_{\nu\geq 1} \|T_\nu\|$, we have for every $x \in R$

$$\|T_\nu S_\nu x - TSx\| \leq \gamma \|S_\nu x - Sx\| + \|T_\nu Sx - TSx\|,$$

and hence we obtain by assumption $\lim_{\nu\to\infty} T_\nu S_\nu x = TSx$ for every $x \in R$.

For an operator T on R and a manifold A of R, we denote by TA the manifold $\{ Tx : x \in A \}$. With this notation, we have

Theorem 10.5. Let B be the closed linear manifold generated by a manifold A of R. For a bounded linear operator T on R, TB is included in the closed linear manifold generated by TA.

Proof. Denoting by A_1 the linear manifold generated by A, it is evident that TA_1 is included in the closed linear manifold generated by TA. Since A_1 is dense in B by Theorem 2.5, for any $\xi \in B$, we can find $a_\nu \in A_1 (\nu = 1, 2, \ldots)$ such that $\lim_{\nu \to \infty} a_\nu = \xi$. Then we have $\lim_{\nu \to \infty} Ta_\nu = T\xi$, because T is bounded by assumption. Therefore TB is included in the closed linear manifold generated by TA.

Finally we consider operators on a Hilbert space. Let R be a Hilbert space in the sequel. For a bounded linear operator T on R, corresponding to every $a \in R$, putting

$$\varphi_a(x) = (a, Tx) \qquad (x \in R),$$

we obtain an anti-linear functional φ_a on R, and for every $x \in R$

$$|\varphi_a(x)| \leqq \|a\| \|Tx\| \leqq (\|a\| \|T\|) \|x\|.$$

Thus we can find by Theorem 5.4 uniquely an element $\xi_a \in R$ such that

$$(a, Tx) = (\xi_a, x) \qquad \text{for every } x \in R.$$

Thus, putting $Sa = \xi_a$ $(a \in R)$, we obtain an operator S on R. This operator S is called the adjoint operator of T and denoted by T^*. Therefore we have

$$(6) \qquad (T^*x, y) = (x, Ty) \qquad \text{for every } x, y \in R.$$

The adjoint operator T^* of T is linear, because we have

$$(T^*(\alpha x + \beta y), z) = (\alpha x + \beta y, Tz) = \alpha(x, Tz) + \beta(y, Tz)$$
$$= \alpha(T^*x, z) + \beta(T^*y, z) = (\alpha T^*x + \beta T^*y, z)$$

for every $z \in R$, and hence by Theorem 7.1

$$T^*(\alpha x + \beta y) = \alpha T^*x + \beta T^*y$$

for every $x, y \in R$ and complex numbers α, β. Furthermore T^* is bounded and we have

$$(7) \qquad \|T^*\| = \|T\|.$$

Because we have by the formula (3) for every $x \in R$

$$\|T^*x\|^2 = (x, TT^*x) \leqq \|x\| \|TT^*x\| \leqq \|x\| \|T\| \|T^*x\|$$

Thus T^* is bounded and $\|T^*\| \leqq \|T\|$. Consequently we have ob-

viously by the definition (6)

$$(8) \qquad (T^*)^* = T,$$

and hence we have

$$\| T \| = \| (T^*)^* \| \leqq \| T^* \|,$$

as proved just above. Therefore we have $\| T \| = \| T^* \|$.

Concerning adjoint operators, we have obviously by definition

$$(9) \qquad (\alpha T + \beta S)^* = \bar{\alpha} T^* + \bar{\beta} S^*,$$

$$(10) \qquad (T S)^* = S^* T^*.$$

Theorem 10.6. *In a Hilbert space $\mathfrak{R}$, for a system of bounded linear operators T_λ $(\lambda \in \Lambda)$ on $\mathfrak{R}$, if $\sup_{\lambda \in \Lambda} \| T_\lambda x \| < +\infty$ for every $x \in \mathfrak{R}$, then we have $\sup_{\lambda \in \Lambda} \| T_\lambda \| < +\infty$.*

Proof. For the adjoint operators T_λ^* of T_λ $(\lambda \in \Lambda)$, we have by the formula §4(2) for every $y \in \mathfrak{R}$

$$\sup_{\|x\| \leqq 1, \lambda \in \Lambda} |(T_\lambda^* x, y)| = \sup_{\|x\| \leqq 1, \lambda \in \Lambda} |(x, T_\lambda y)| \leqq \sup_{\lambda \in \Lambda} \| T_\lambda y \| < +\infty.$$

Thus we obtain by Theorem 9.12 $\sup_{\|x\| \leqq 1, \lambda \in \Lambda} \| T_\lambda^* x \| < +\infty$, and hence

$$\sup_{\lambda \in \Lambda} \| T_\lambda \| = \sup_{\lambda \in \Lambda} \| T_\lambda^* \| = \sup_{\lambda \in \Lambda} \left\{ \sup_{\|x\| \leqq 1} \| T_\lambda^* x \| \right\} < +\infty.$$

Recalling Theorem 10.3, we obtain by this Theorem 10.6

Theorem 10.7. *For a sequence of bounded linear operators T_ν $(\nu = 1, 2, \ldots)$ on a Hilbert space $\mathfrak{R}$, if $T = \lim_{\nu \to \infty} T_\nu$, then T is a bounded linear operator on $\mathfrak{R}$, and $\sup_{\nu} \| T_\nu \| < +\infty$.*

Furthermore we conclude from Theorem 10.4

Theorem 10.8. *In a Hilbert space $\mathfrak{R}$, for bounded linear operators T_ν, S_ν $(\nu = 1, 2, \ldots)$, if $\lim_{\nu \to \infty} T_\nu = T$, $\lim_{\nu \to \infty} S_\nu = S$, then we have $\lim_{\nu \to \infty} T_\nu S_\nu = TS$.*

Theorem 10.9. *For a bounded linear operator T, $\mathrm{w}\text{-}\lim_{\nu \to \infty} a_\nu = a$ implies $\mathrm{w}\text{-}\lim_{\nu \to \infty} T a_\nu = T a$.*

Proof. For the adjoint operator T^* of T, if $\mathrm{w}\text{-}\lim_{\nu \to \infty} a_\nu = a$, then we have for every $x \in \mathfrak{R}$

$$\lim_{\nu \to \infty} (T a_\nu, x) = \lim_{\nu \to \infty} (a_\nu, T^* x) = (a, T^* x) = (T a, x),$$

and hence $\underset{\nu\to\infty}{\text{w-lim}}\, T a_\nu = Ta$.

An operator T on a Hilbert space R is said to be symmetric, if

$$(Tx, y) = (x, Ty) \qquad \text{for every } x, y \in R.$$

With this definition, we have

Theorem 10.10. Every symmetric operator T on a Hilbert space R is a bounded linear operator and $T^* = T$.

Proof. Since we have obviously by assumption for every $z \in R$

$$(T(\alpha x + \beta y), z) = (\alpha x + \beta y, Tz)$$

$$= \alpha(x, Tz) + \beta(y, Tz) = (\alpha Tx + \beta Ty, z),$$

we obtain by Theorem 7.1

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

for every $x, y \in R$ and complex numbers α, β. Since we have by the formula §4(2)

$$\sup_{\|x\|\leq 1} |(Tx, y)| = \sup_{\|x\|\leq 1} |(x, Ty)| \leqq \|Ty\|,$$

we obtain by Theorem 9.12 $\sup_{\|x\|\leq 1} \|Tx\| < +\infty$, and hence T is bounded. It is obvious by definition that $T^* = T$.

Theorem 10.11. If a linear operator T on a Hilbert space R satisfies $(Tx, x) = (x, Tx)$ for every $x \in R$, then T is a symmetric operator on R.

Proof. We have by assumption for every complex number λ

$$(T(x + \lambda y), x + \lambda y) = (x + \lambda y, T(x + \lambda y)),$$

$$(Tx, x) = (x, Tx), \qquad (Ty, y) = (y, Ty).$$

Since T is linear by assumption, we obtain then

$$\lambda(Ty, x) + \bar{\lambda}(Tx, y) = \lambda(y, Tx) + \bar{\lambda}(x, Ty).$$

Thus, putting $\lambda = 1$, we have

$$(Ty, x) + (Tx, y) = (y, Tx) + (x, Ty),$$

and, putting $\lambda = i$, we obtain

$$(Ty, x) - (Tx, y) = (y, Tx) - (x, Ty).$$

Therefore we conclude $(Ty, x) = (y, Tx)$ for every $x, y \in R$.

CHAPTER II

SPECTRAL THEORY

§11 Projection Operators

Let R be a Hilbert space in the sequel. For an arbitrary manifold A of R, the manifold $\{x : x \perp A\}$ is called the orthogonal complement of A and denoted by $A^{\perp}$. With this definition, we have obviously

(1) $R^{\perp} = \{0\}, \quad \{0\}^{\perp} = R,$

(2) $A \supset B$ implies $A^{\perp} \subset B^{\perp},$

(3) $A \subset A^{\perp\perp},$

(4) $AA^{\perp} \subset \{0\}.$

Since $A \subset A^{\perp\perp}$ by (3), we obtain $A^{\perp} \supset (A^{\perp\perp})^{\perp}$ by (2). On the other hand we have $A^{\perp} \subset (A^{\perp})^{\perp\perp}$ by (3). Therefore we conclude

(5) $A^{\perp} = A^{\perp\perp\perp}.$

Furthermore we see easily by definition for a system A_{λ} $(\lambda \in \Lambda)$

(6) $\prod_{\lambda\in\Lambda} A_{\lambda}^{\perp} = \{\sum_{\lambda\in\Lambda} A_{\lambda}\}^{\perp} = \{\sum_{\lambda\in\Lambda} A_{\lambda}^{\perp\perp}\}^{\perp}.$

Recalling the relations (2), (3), (5) in §7, we see easily

Theorem 11.1. The orthogonal complement $A^{\perp}$ is a closed linear manifold for every manifold A.

We have further

Theorem 11.2. $A^{\perp\perp}$ is the closed linear manifold generated by A for every manifold A.

Proof. Let B be the closed linear manifold generated by A. Since $A^{\perp\perp}$ is a closed linear manifold by Theorem 11.1 and $A^{\perp\perp} \supset A$ by (3), we have $B \subset A^{\perp\perp}$ by definition, and hence $B^{\perp} \supset A^{\perp\perp\perp} = A^{\perp}$ by (2), (5). On the other hand, $B \supset A$ implies $B^{\perp} \subset A^{\perp}$ by (2). We have therefore $B^{\perp} = A^{\perp}$, and consequently $B^{\perp\perp} = A^{\perp\perp}$. Furthermore B is complete by Theorem 2.9, because R is complete by assumption. Thus for any $\alpha \in B$ we can find by Theorem 7.2 $\xi \in R$ such that $\|\xi\| = 1$, $\xi \perp B$ and $\xi = \alpha\alpha + \chi$ for some $\chi \in B$ and complex number α. Here

we have $\alpha \neq 0$, because $0 \neq \xi \perp B$ implies $\xi \in B$ by (4). Hence we have $a = -x + \frac{1}{\alpha}\xi$, $-x \in B$, $\frac{1}{\alpha}\xi \perp B$. Therefore we can state that for any $a \in R$ we can find $x \in B$ and $y \in B^{\perp}$ such that $a = x + y$. Especially for $a \in B^{\perp\perp}$, we have then $B^{\perp\perp} \ni a - x = y \in B^{\perp}$, because $B \subset B^{\perp\perp}$ by (3). Thus we conclude $y = 0$ by (4), and hence $a = x \in B$. Consequently we have $B = B^{\perp\perp} = A^{\perp\perp}$.

In this Proof, we have proved that for any $a \in R$ we can find $x \in B$ and $y \in B^{\perp}$ such that $a = x + y$. Such x, y are determined uniquely. Because, if

$$x + y = x_1 + y_1, \qquad x, x_1 \in B; \quad y, y_1 \in B^{\perp},$$

then we have $B \ni x - x_1 = y_1 - y \in B^{\perp}$, and hence $x - x_1 = y_1 - y = 0$ by (4). Therefore we can state

Theorem 11.3. *If A is a closed linear manifold of R, then for any $a \in R$ we can find uniquely $x \in A$ and $y \in A^{\perp}$ such that*

$$a = x + y.$$

Let M be a closed linear manifold of R in the sequel. For any $a \in R$ we can find by Theorem 11.3 uniquely $x \in M$ and $y \in M^{\perp}$ such that $a = x + y$. This uniquely determined element x is called the projection of an element a to M and denoted by $P_M a$. Then we obtain obviously an operator P_M on R. This operator P_M is called the projection operator of a closed linear operator M.

Since $M = M^{\perp\perp}$ by Theorem 11.2, we have obviously by definition

$$P_M + P_{M^{\perp}} = 1, \tag{7}$$

for the projection operator $P_{M^{\perp}}$ of $M^{\perp}$. Recalling the formula (1), we conclude immediately by definition

$$P_R = 1, \qquad P_{\{0\}} = 0. \tag{8}$$

It is evident by definition that $P_M x \in M$ for every $x \in R$. Thus $P_M x = x$ implies $x \in M$. Conversely $x \in M$ implies $x = P_M x$ by definition, because $x = x + 0$, $x \in M$, $0 \in M^{\perp}$. Therefore we can state

Theorem 11.4. *We have $P_M x = x$, if and only if $x \in M$.*

We see at once by the formula (7) that $P_M x = 0$ if and only if $P_{M^{\perp}} x = x$. Therefore we have by Theorem 11.4

Theorem 11.5. We have $P_M x = 0$ if and only if $x \in M^\perp$.

Since $P_M x \in M$ for every $x \in R$, we obtain by Theorem 11.5

$$P_M P_M x = P_M x \qquad \text{for every } x \in R.$$

Thus every projection operator P_M is idenpotent:

(9) $$P_M P_M = P_M.$$

Furthermore we can state

Theorem 11.6. The range of P_M coincides with M, that is,

$$P_M R = M.$$

Since $P_M x \in M$, $P_{M^\perp} y \in M^\perp$ for every x, $y \in R$, we have

$$(P_M x, P_{M^\perp} y) = 0 \qquad \text{for every } x, y \in R.$$

This relation yields by the formula (7)

$$(P_M x, y) = (P_M x, P_M y).$$

We also obtain likewise

$$(x, P_M y) = (P_M x, P_M y).$$

Therefore we have

(10) $$(P_M x, y) = (x, P_M y) = (P_M x, P_M y).$$

From this relation we conclude that the projection operator P_M is symmetric. Therefore P_M is a bounded linear operator on R by Theorem 10.9, and hence P_M is continuous. Furthermore we have

(11) $$\| P_M x \| \leqq \| x \| \qquad \text{for every } x \in R.$$

Because, recalling the formula §7(4), we obtain by (7)

$$\| x \|^2 = \| P_M x \|^2 + \| P_{M^\perp} x \|^2.$$

This relation yields (11) at once, and we obtain further by (7)

(12) $$\| (1 - P_M) x \|^2 = \| x \|^2 - \| P_M x \|^2.$$

Furthermore we have $\| P_{M^\perp} x \| = 0$ by Theorem 11.4, if and only if $x \in M^{\perp\perp} = M$. Therefore we have

Theorem 11.7. $\| P_M x \| = \| x \|$ implies $P_M x = x$.

By virtue of the formulas (9), (10), every projection operator is a symmetric idenpotent operator on R. Conversely we have

Theorem 11.8. For a symmetric idenpotent operator T on R, can find uniquely a closed linear manifold M such that $T = P_M$.

Proof. Since T is symmetric and idenpotent by assumption, we have for every x, $y \in R$

$$(Tx, y - Ty) = (x, Ty - TTy) = 0.$$

Therefore, denoting by A the range of T, we have for every $x \in R$

$$x = Tx + (x - Tx), \quad Tx \in A, \quad x - Tx \in A^{\perp}$$

Since $A \subset A^{\perp\perp}$ by (3), we conclude hence by definition

$$x - Tx = P_{A^{\perp}} x.$$

This relation yields by (7) $T = P_{A^{\perp\perp}}$.

By virtue of this Theorem 11.8, every symmetric idenpotent operator on R also is called a projection operator.

§12 Calculus of Projection Operators

For two closed linear manifolds M and N of R, we define $P_M \leqq P_N$ or $P_N \geqq P_M$ to mean $M \subset N$. With this definition, we have obviously by the formula §11(2) that $P_M \leqq P_N$ is equivalent to $P_{M^{\perp}} \geqq P_{N^{\perp}}$, and hence by §11(7) that $P_M \leqq P_N$ is equivalent to $1 - P_M \geqq 1 - P_N$.

Theorem 12.1. *For two projection operators P and Q, $P \leqq Q$ is equivalent to every one of the following conditions:*

1) $QP = P$,

2) $PQ = P$,

3) $\|Qx - Px\|^2 = \|Qx\|^2 - \|Px\|^2$ *for every* $x \in R$,

4) $\|Px\| \leqq \|Qx\|$ *for every* $x \in R$,

5) $Qx = 0$ *implies* $Px = 0$.

Proof. Recalling Theorems 11.4 and 11.6, we see easily by definition that $P \leqq Q$ implies $QPx = Px$ for every $x \in R$, that is, $P \leqq Q$ implies the relation 1).

If $QP = P$, then we have by the formula §10(10)

$$P = P^* = (QP)^* = P^*Q^* = PQ,$$

because every projection operator is symmetric. Thus 1) implies 2).

If $P = PQ$, then we have by the formulas (9), (10) in §11

$$(Px, Qx - Px) = (x, PQx - Px) = 0,$$

and hence by §7(4)

$$\|Qx\|^2 = \|Px\|^2 + \|Qx - Px\|^2 \quad \text{for every } x \in R.$$

Thus 2) implies 3). It is evident that 3) implies 4), and that 4) implies 5). Recalling Theorem 11.5 and the formula §11(2), we see easily by definition that 5) implies $P \leqq Q$. Therefore our assertion is established.

For two manifolds A and B of R, we define $A \perp B$ to mean $A \subset B^{\perp}$, that is, $x \perp y$ for every $x \in A$ and $y \in B$. With this definition, it is evident that $A \perp B$ implies $B \perp A$.

Theorem 12.2. *For two closed linear manifolds M and N, we have $M \perp N$ if and only if $P_M P_N = 0$.*

Proof. If $M \perp N$, then, since $P_M x \in M$, $P_N y \in N$ by definition, we have $(P_M x, P_N y) = 0$, and hence by §11(10)

$$(x, P_M P_N y) = 0 \qquad \text{for every } x, y \in R.$$

Thus we conclude $P_M P_N = 0$ by Theorem 7.1.

Conversely, $P_M P_N = 0$ implies by §11(10) for every $x, y \in R$

$$(P_M x, P_N y) = (x, P_M P_N y) = 0,$$

and hence $M \perp N$ by Theorem 11.6.

Theorem 12.3. *For two projection operators P and Q, PQ also is a projection operator if and only if $PQ = QP$.*

Proof. If PQ is a projection operator, then PQ is symmetric, and hence by the formula §10(10)

$$PQ = (PQ)^* = Q^* P^* = QP.$$

Conversely if $PQ = QP$, then we have by §11(10)

$$(PQx, y) = (x, QPy) = (x, PQy),$$

and hence PQ is symmetric. Furthermore we have

$$(PQ)(PQ) = (QP)(PQ) = QPQ = PQQ = PQ,$$

that is, PQ is idenpotent. Therefore PQ is a projection operator by Theorem 11.8.

Theorem 12.4. *For two projection operators P and Q, $P+Q$ also is a projection operator if and only if $PQ = 0$.*

Proof. If $P+Q$ is a projection operator, then $P+Q$ is idenpotent, and hence

$$P+Q = (P+Q)(P+Q) = P+Q+PQ+QP.$$

This relation yields

$$PQ + QP = 0, \qquad PQ + QPQ = 0,$$

and further $QPQ + QPQ = 0$. Thus we obtain $QPQ = 0$, and consequently $PQ = 0$.

Conversely if $PQ = 0$, then we have $QP = 0$ by Theorem 12.2, and hence

$$(P + Q)(P + Q) = P + Q,$$

that is, $P + Q$ is idenpotent. Since $P + Q$ is symmetric by §10(9), $P + Q$ is a projection operator by Theorem 11.8.

Theorem 12.5. For two closed linear manifolds M and N of R, $P_M - P_N$ is a projection operator if and only if $P_M \geqq P_N$, and then

$$P_M - P_N = P_{MN^\perp}.$$

Proof. If $P_M - P_N$ is a projection operator, then we have by Theorem 12.4

$$P_N(P_M - P_N) = 0,$$

because $P_M = P_N + (P_M - P_N)$. Thus we have

$$P_N P_M = P_N P_N = P_N,$$

and hence $P_N \leqq P_M$ by Theorem 12.1.

Conversely if $P_N \leqq P_M$, then $P_M P_N = P_N P_M = P_N$ by Theorem 12.1, and consequently

$$(P_M - P_N)(P_M - P_N) = P_M P_M + P_N P_N - P_M P_N - P_N P_M = P_M - P_N,$$

that is, $P_M - P_N$ is idenpotent. Since $P_M - P_N$ is obviously symmetric, $P_M - P_N$ is a projection operator by Theorem 11.8. Furthermore

$$P_M(P_M - P_N) = P_M - P_N,$$

$$P_{N^\perp}(P_M - P_N) = (1 - P_N)(P_M - P_N) = P_M - P_N.$$

Thus we see by Theorem 11.4 that the range of $P_M - P_N$ is included in $MN^\perp$. On the other hand, for any $x \in MN^\perp$, we have by Theorems 11.4 and 11.5 $P_M x = x$, $P_N x = 0$, and hence $(P_M - P_N)x = x$. Therefore we conclude by Theorem 11.4 that the range of $P_M - P_N$ coincides with $MN^\perp$, and hence $P_M - P_N = P_{MN^\perp}$ by Theorem 11.6.

For two closed linear manifolds M and N, the projection operator of MN will be denoted by $P_M \frown P_N$, that is

$$P_{MN} = P_M \frown P_N.$$

On the other hand, we shall denote by $P_M \smile P_N$ the projection operator

of the closed linear manifold generated by M and N. With this definition, we have obviously for every projection operators P and Q

$$P \cup Q = Q \cup P \geqq P, \qquad P \cap Q = Q \cap P \leqq P.$$

Furthermore, for projection operators P_1, P_2, P_3, it is obvious that P_1, $P_2 \leqq P_3$ implies $P_1 \cup P_2 \leqq P_3$, and that P_1, $P_2 \geqq P_3$ implies $P_1 \cap P_2 \geqq P_3$.

Theorem 12.6. *For two closed linear manifolds M and N, we have $P_M P_N = P_N P_M$ if and only if $P_N M \subset M$, and then*

$$P_N M = NM, \qquad P_M \cap P_N = P_M P_N,$$

$$P_M \cup P_N = P_M + P_N - P_M P_N.$$

Proof. If $P_M P_N = P_N P_M$, then for every $x \in M$ we have by Theorem 11.4 $P_N x = P_N P_M x = P_M P_N x \in M$.

Conversely if $P_N M \subset M$, then, since $P_M x \in M$ for every $x \in R$ by Theorem 11.6, we have by Theorem 11.4 for every $x \in R$

$$P_M P_N P_M x = P_N P_M x,$$

that is, $P_M P_N P_M = P_N P_M$. Thus we obtain by §10(10)

$$P_M P_N = P_M^* P_N^* = (P_N P_M)^* = (P_M P_N P_M)^* = P_M P_N P_M = P_N P_M.$$

If $P_M P_N = P_N P_M$, then $P_M P_N$ is by Theorem 11.8 a projection operator, and we have obviously

$$P_M P_N x = P_N P_M x \in MN \qquad \text{for every } x \in R.$$

On the other hand, for every $x \in MN$, we have by Theorem 11.4

$$P_M P_N x = P_M x = x.$$

Therefore we see by Theorem 11.6 that

$$P_M P_N = P_{MN} \qquad \text{and} \qquad P_N M = NM.$$

Denoting by A the closed linear manifold generated by M and N, we have by Theorem 11.2 and the formula §11(6)

$$A = (M^\perp N^\perp)^\perp.$$

Since we have by the formula §11(7)

$$P_{M^\perp} P_{N^\perp} = (1 - P_M)(1 - P_N) = (1 - P_N)(1 - P_M) = P_{N^\perp} P_{M^\perp},$$

we have $P_{M^\perp} P_{N^\perp} = P_{M^\perp N^\perp}$, as proved just above. Therefore we obtain by the formula §11(7)

$$P_M \cup P_N = P_A = 1 - P_{M^\perp N^\perp} = 1 - P_{M^\perp} P_{N^\perp}$$
$$= 1 - (1 - P_M)(1 - P_N) = P_M + P_N - P_M P_N.$$

Theorem 12.7. *For two projection operators P and Q, if they*

are commutative: $PQ = QP$, then we have for every $x \in R$

$$\| P \vee Q\, x \|^2 \leqq \| P x \|^2 + \| Q x \|^2 .$$

Proof. Recalling the formulas §11(11) and §7(4), we have by Theorem 12.6 for every $x \in R$

$$\| P \vee Q\, x \|^2 = \| P x + Q(1-P)x \|^2$$
$$= \| P x \|^2 + \| (1-P) Q x \|^2 \leqq \| P x \|^2 + \| Q x \|^2$$

For an operator T on R, we define T^ν $(\nu = 1, 2, \ldots)$ to mean

$$T^1 = T, \qquad T^{\nu+1} = T T^\nu \qquad (\nu = 1, 2, \ldots).$$

With this notation we have

Theorem 12.8. For two closed linear manifolds M, N, we have

$$P_M \wedge P_N = P_{MN} = \lim_{\nu \to \infty} (P_M P_N)^\nu .$$

Proof. Putting $T = P_N P_M P_N$, we obtain by §10(10) a symmetric operator T on R, and we have by §11(11)

$$\| T x \| \leqq \| x \| \qquad \text{for every } x \in R .$$

For an arbitrary $a \in R$, we have hence

$$(T^{2\nu} a, T^{2\mu} a) = \| T^{\mu+\nu} a \|^2 ,$$

and consequently $(T^{2\nu} a, T^{2\mu} a)$ $(\nu = 1, 2, \ldots)$ is convergent for every $\mu = 1, 2, \ldots$. Therefore we can find by Theorem 9.10 $f \in R$ such that $w\text{-}\lim_{\nu \to \infty} T^{2\nu} a = f$. For such $f \in R$ we have

$$\| f \|^2 = \lim_{\nu \to \infty} (T^{2\nu} a, f) = \lim_{\nu \to \infty} \lim_{\mu \to \infty} (T^{2\nu} a, T^{2\mu} a)$$
$$= \lim_{\nu \to \infty} \lim_{\mu \to \infty} (a, T^{2(\mu+\nu)} a) = (a, f)$$
$$= \lim_{\nu \to \infty} (a, T^{2\nu} a) = \lim_{\nu \to \infty} \| T^\nu a \|^2 .$$

Therefore we have $\lim_{\nu \to \infty} T^{2\nu} a = f$ by Theorem 9.5. Furthermore, putting $Q = \lim_{\nu \to \infty} T^{2\nu}$, we obtain by Theorem 10.6 a bounded linear operator Q on R. This operator Q is symmetric, because

$$(Q x, y) = \lim_{\nu \to \infty} (T^{2\nu} x, y) = \lim_{\nu \to \infty} (x, T^{2\nu} y) = (x, Q y);$$

and Q is idenpotent, because we have by Theorem 10.8

$$Q Q = \lim_{\nu \to \infty} T^{4\nu} = Q .$$

Therefore Q is a projection operator by Theorem 11.8. Since

$$T^2 Q x = \lim_{\nu \to \infty} T^{2\nu+2} x = Q x ,$$

we have for every $x \in R$ by §11(11)

$$\| Q x \| \leqq \| T Q x \| \leqq \| P_M P_N Q x \| \leqq \| P_N Q x \| \leqq \| Q x \| ,$$

and hence $\| Q x \| = \| P_N Q x \| = \| P_M P_N Q x \|$. Thus we conclude by

Theorem 11.7

$$Q x = P_N Q x = P_M P_N Q x \quad \text{for every } x \in R,$$

and hence the range of Q is included in MN by Theorem 11.4. On the other hand, for every $x \in MN$, we have by Theorem 11.4

$$Tx = P_N P_M P_N x = x.$$

Therefore we conclude $Q = P_{MN}$ by Theorem 11.6.

Since $TQ = P_N P_M P_N Q = Q$, we also have $Q = \lim_{\nu \to \infty} T^{2\nu+1}$, and hence $Q = \lim_{\nu \to \infty} T^{\nu}$. On the other hand we have obviously by the formula §11(9)

$$(P_M P_N)^{\nu+1} = P_M T^{\nu} \qquad (\nu = 1, 2, \ldots).$$

Thus we obtain further

$$Q = P_M Q = \lim_{\nu \to \infty} P_M T^{\nu} = \lim_{\nu \to \infty} (P_M P_N)^{\nu}.$$

§13 Systems of Projection Operators

For a system of closed linear manifolds M_λ $(\lambda \in \Lambda)$ of R, the intersection $\prod_{\lambda \in \Lambda} M_\lambda$ also is obviously a closed linear manifold of R. The projection operator of $\prod_{\lambda \in \Lambda} M_\lambda$ will be denoted by $\bigwedge_{\lambda \in \Lambda} P_{M_\lambda}$, i.e.

$$(1) \qquad \bigwedge_{\lambda \in \Lambda} P_{M_\lambda} = P_{\prod_{\lambda \in \Lambda} M_\lambda}$$

With this definition we have obviously by Theorem 11.4

Theorem 13.1. *In order that* $Q = \bigwedge_{\lambda \in \Lambda} P_\lambda$, *it is necessary and sufficient that we have* $Q x = x$ *if and only if* $P_\lambda x = x$ *for every* $\lambda \in \Lambda$.

For a system of projection operators P_λ $(\lambda \in \Lambda)$, we shall write $P_\lambda \downarrow_{\lambda \in \Lambda}$, if for any two $\lambda_1, \lambda_2 \in \Lambda$ we can find $\lambda \in \Lambda$ such that $P_\lambda \leqq P_{\lambda_1} \wedge P_{\lambda_2}$; and $P_\lambda \downarrow_{\lambda \in \Lambda} Q$ means that $P_\lambda \downarrow_{\lambda \in \Lambda}$ and $Q = \bigwedge_{\lambda \in \Lambda} P_\lambda$. For a sequence of projection operators P_ν $(\nu = 1, 2, \ldots)$, we shall write $P_\nu \downarrow_{\nu=1}^{\infty}$ if $P_1 \geqq P_2 \geqq \cdots$; and $P_\nu \downarrow_{\nu=1}^{\infty} Q$ means $P_\nu \downarrow_{\nu=1}^{\infty}$ and $Q = \bigwedge_{\nu=1}^{\infty} P_\nu$

Theorem 13.2. *If* $P_\lambda \downarrow_{\lambda \in \Lambda} Q$, *then we have*

$$\|Q x\| = \inf_{\lambda \in \Lambda} \|P_\lambda x\| \qquad \text{for every } x \in R,$$

and $P_{\lambda_\nu} \downarrow_{\nu=1}^{\infty}$, $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$, $\lim_{\nu \to \infty} \|P_{\lambda_\nu} x\| = \|Q x\|$ *implies* $\lim_{\nu \to \infty} P_{\lambda_\nu} x = Q x$.

Proof. For an arbitrary $a \in R$, putting $\gamma = \inf_{\lambda \in \Lambda} \| P_\lambda a \|$, we can find by assumption $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ such that

$$P_{\lambda_\nu} \downarrow_{\nu=1}^{\infty}, \qquad \lim_{\nu \to \infty} \| P_{\lambda_\nu} a \| = \gamma.$$

For such $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$, $(P_{\lambda_\nu} - P_{\lambda_{\nu+1}})a$ $(\nu = 1, 2, \ldots)$ is an orthogonal sequence, because we have by Theorem 12.1 for $\nu < \mu$

$$((P_{\lambda_\nu} - P_{\lambda_{\nu+1}})a, (P_{\lambda_\mu} - P_{\lambda_{\mu+1}})a) = ((P_{\lambda_\mu} - P_{\lambda_{\mu+1}})(P_{\lambda_\nu} - P_{\lambda_{\nu+1}})a, a) = 0.$$

Furthermore we have by the formula §7(4)

$$\sum_{\nu=1}^{\infty} \| (P_{\lambda_\nu} - P_{\lambda_{\nu+1}})a \|^2 = \lim_{\nu \to \infty} \| (P_{\lambda_1} - P_{\lambda_\nu})a \|^2 \leqq \| a \|^2.$$

Therefore $\sum_{\nu=1}^{\infty} (P_{\lambda_\nu} - P_{\lambda_{\nu+1}})a$ is convergent by Theorem 7.11, and hence $P_{\lambda_\nu} a$ $(\nu = 1, 2, \ldots)$ is convergent. Thus we can put

$$\xi = \lim_{\nu \to \infty} P_{\lambda_\nu} a$$

and we have obviously

$$\| \xi \| = \lim_{\nu \to \infty} \| P_{\lambda_\nu} a \| = \gamma.$$

For any $\lambda \in \Lambda$ we can find by assumption $\rho_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ such that $P_{\rho_\nu} \leqq P_\lambda \cap P_{\lambda_\nu}$, and then we have

$$\| P_\lambda \xi \| = \lim_{\nu \to \infty} \| P_\lambda P_{\lambda_\nu} a \| \geqq \overline{\lim_{\nu \to \infty}} \| P_{\rho_\nu} a \| \geqq \gamma = \| \xi \|.$$

Consequently $\| P_\lambda \xi \| = \| \xi \|$ by §11(11), and hence we obtain by Theorem 11.7 $P_\lambda \xi = \xi$ for every $\lambda \in \Lambda$. Thus we have $Q \xi = \xi$ by Theorem 13.1. On the other hand, since $Q \leqq P_{\lambda_\nu}$ $(\nu = 1, 2, \ldots)$, we have by Theorem 12.1

$$Q \xi = \lim_{\nu \to \infty} Q P_{\lambda_\nu} a = Q a.$$

Therefore $Q a = \xi = \lim_{\nu \to \infty} P_{\lambda_\nu} a$, and further $\| Q a \| = \gamma$.

For a system of closed linear manifolds M_λ $(\lambda \in \Lambda)$, the closed linear manifold generated by $\sum_{\lambda \in \Lambda} M_\lambda$ coincides with $(\sum_{\lambda \in \Lambda} M_\lambda)^{\perp\perp}$ by Theorem 11.2. The projection operator of $(\sum_{\lambda \in \Lambda} M_\lambda)^{\perp\perp}$ will be denoted by $\bigcup_{\lambda \in \Lambda} P_{M_\lambda}$, that is,

$$(2) \qquad \bigcup_{\lambda \in \Lambda} P_{M_\lambda} = P_{(\sum_{\lambda \in \Lambda} M_\lambda)^{\perp\perp}}.$$

With this definition we have obviously by Theorem 11.5

Theorem 13.3. *In order that* $Q = \bigcup_{\lambda \in \Lambda} P_\lambda$, *it is necessary and sufficient that we have* $Q x = 0$ *if and only if* $P_\lambda x = 0$ *for all* $\lambda \in \Lambda$.

Since $(\sum_{\lambda \in \Lambda} M_\lambda)^{\perp} = \prod_{\lambda \in \Lambda} M_\lambda^{\perp}$ by §11(6), we obtain by §11(7)

$$(3) \qquad \bigcup_{\lambda \in \Lambda} P_\lambda = 1 - \bigcap_{\lambda \in \Lambda} (1 - P_\lambda).$$

From this relation we conclude further

$$(4) \qquad \bigwedge_{\lambda\in\Lambda} P_\lambda = 1 - \bigvee_{\lambda\in\Lambda} (1 - P_\lambda).$$

For a system of projection operators P_λ $(\lambda \in \Lambda)$, we shall write $P_\lambda \uparrow_{\lambda\in\Lambda}$, if for any λ_1, $\lambda_2 \in \Lambda$ we can find $\lambda \in \Lambda$ such that $P_\lambda \geqq P_{\lambda_1} \vee P_{\lambda_2}$; and $P_\lambda \uparrow_{\lambda\in\Lambda} Q$ means that $P_\lambda \uparrow_{\lambda\in\Lambda}$ and $Q = \bigvee_{\lambda\in\Lambda} P_\lambda$. For a sequence of projection operators P_ν $(\nu = 1, 2, \ldots)$ we shall write $P_\nu \uparrow_{\nu=1}^{\infty}$, if $P_1 \leqq P_2 \leqq \cdots$; and $P_\nu \uparrow_{\nu=1}^{\infty} Q$ means that $P_\nu \uparrow_{\nu=1}^{\infty}$ and $Q = \bigvee_{\nu=1} P_\nu$.

Theorem 13.4. *If $P_\lambda \uparrow_{\lambda\in\Lambda} Q$, then we have*

$$\| Q x \| = \sup_{\lambda\in\Lambda} \| P_\lambda x \| \qquad \text{for every } x \in R,$$

and $P_{\lambda_\nu} \uparrow_{\nu=1}^{\infty}$, $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$, $\lim_{\nu\to\infty} \| P_{\lambda_\nu} x \| = \| Q x \|$ implies $\lim_{\nu\to\infty} P_{\lambda_\nu} x = Q x$.

Proof. If $P_\lambda \uparrow_{\lambda\in\Lambda} Q$, then we have by the formula (3)

$$(1 - P_\lambda) \downarrow_{\lambda\in\Lambda} 1 - Q.$$

Thus we have by Theorem 13.2 for every $x \in R$

$$\| x - Q x \|^2 = \inf_{\lambda\in\Lambda} \| x - P_\lambda x \|^2$$

and hence by §7(4)

$$\| x \|^2 - \| Q x \|^2 = \inf_{\lambda\in\Lambda} (\| x \|^2 - \| P_\lambda x \|^2).$$

This relation yields $\| Q x \|^2 = \sup_{\lambda\in\Lambda} \| P_\lambda x \|^2$, and hence

$$\| Q x \| = \sup_{\lambda\in\Lambda} \| P_\lambda x \| \qquad \text{for every } x \in R.$$

If $P_{\lambda_\nu} \uparrow_{\nu=1}^{\infty}$ and $\lim_{\nu\to\infty} \| P_{\lambda_\nu} x \| = \| Q x \|$, then we have

$$(1 - P_{\lambda_\nu}) \downarrow_{\nu=1}^{\infty}, \qquad \lim_{\nu\to\infty} \| x - P_{\lambda_\nu} x \| = \| x - Q x \|,$$

and hence by Theorem 13.2

$$\lim_{\nu\to\infty} (x - P_{\lambda_\nu} x) = x - Q x,$$

that is, $\lim_{\nu\to\infty} P_{\lambda_\nu} x = Q x$.

By virtue of Theorems 13.2 and 13.4, we have obviously

Theorem 13.5. *$P_\nu \uparrow_{\nu=1}^{\infty} Q$ or $P_\nu \downarrow_{\nu=1}^{\infty} Q$ implies $\lim_{\nu\to\infty} P_\nu = Q$.*

Theorem 13.6. *If a projection operator Q is commutative to all projection operators P_λ $(\lambda \in \Lambda)$, then we have*

$$Q \bigwedge_{\lambda\in\Lambda} P_\lambda = \left(\bigwedge_{\lambda\in\Lambda} P_\lambda\right) Q = \bigwedge_{\lambda\in\Lambda} P_\lambda Q,$$

$$Q \bigvee_{\lambda\in\Lambda} P_\lambda = \left(\bigvee_{\lambda\in\Lambda} P_\lambda\right) Q = \bigvee_{\lambda\in\Lambda} P_\lambda Q.$$

$$Q \cup \bigcap_{\lambda \in \Lambda} P_\lambda = \bigcap_{\lambda \in \Lambda} (Q \cup P_\lambda).$$

Proof. Denoting by M_λ the range of P_λ $(\lambda \in \Lambda)$ and by N that of Q, we have by definition

$$\bigcap_{\lambda \in \Lambda} P_\lambda = P_{\prod_{\lambda \in \Lambda} M_\lambda}, \quad P_\lambda = P_{M_\lambda}, \quad Q = P_N.$$

Since $P_N P_{M_\lambda} = P_{M_\lambda} P_N$ $(\lambda \in \Lambda)$ by assumption, we have by Theorem 12.6 $P_N M_\lambda \subset M_\lambda$ for every $\lambda \in \Lambda$, and hence $P_N \prod_{\lambda \in \Lambda} M_\lambda \subset \prod_{\lambda \in \Lambda} M_\lambda$. Thus we obtain by Theorem 12.6

$$P_N P_{\prod_{\lambda \in \Lambda} M_\lambda} = P_{\prod_{\lambda \in \Lambda} M_\lambda} P_N = P_{\prod_{\lambda \in \Lambda} M_\lambda N}$$
$$= \bigcap_{\lambda \in \Lambda} P_{M_\lambda N} = \bigcap_{\lambda \in \Lambda} P_{M_\lambda} P_N.$$

Therefore the first assertion is established. Consequently we obtain by the formula (3) and Theorem 12.1

$$Q \bigcup_{\lambda \in \Lambda} P_\lambda = (\bigcup_{\lambda \in \Lambda} P_\lambda) Q = Q - \bigcap_{\lambda \in \Lambda} (Q - Q P_\lambda)$$
$$= Q(1 - \bigcap_{\lambda \in \Lambda} (1 - Q P_\lambda)) = \bigcup_{\lambda \in \Lambda} Q P_\lambda,$$

because $Q \geq \bigcup_{\lambda \in \Lambda} Q P_\lambda = 1 - \bigcap_{\lambda \in \Lambda} (1 - Q P_\lambda)$. Furthermore we conclude by the formulas (3) and (4)

$$Q \cup \bigcap_{\lambda \in \Lambda} P_\lambda = 1 - (1 - Q)(1 - \bigcap_{\lambda \in \Lambda} P_\lambda) = 1 - (1 - Q) \bigcup_{\lambda \in \Lambda} (1 - P_\lambda)$$
$$= 1 - \bigcup_{\lambda \in \Lambda} (1 - Q)(1 - P_\lambda) = 1 - \bigcup_{\lambda \in \Lambda} (1 - (Q \cup P_\lambda)) = \bigcap_{\lambda \in \Lambda} (Q \cup P_\lambda).$$

Theorem 13.7. For a system of projection operators P_λ $(\lambda \in \Lambda)$, if $P_\lambda P_\gamma = P_\gamma P_\lambda$ for every $\lambda, \gamma \in \Lambda$, then we have for every $x \in R$

$$\| \bigcup_{\lambda \in \Lambda} P_\lambda x \|^2 \leq \sum_{\lambda \in \Lambda} \| P_\lambda x \|^2.$$

Proof. By virtue of Theorem 12.7, we have for every finite number of elements $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots, \kappa)$

$$\| \bigcup_{\nu=1}^{\kappa} P_{\lambda_\nu} x \|^2 \leq \sum_{\nu=1}^{\kappa} \| P_{\lambda_\nu} x \|^2 \leq \sum_{\lambda \in \Lambda} \| P_\lambda x \|^2,$$

and hence we obtain our assertion by Theorem 13.4, because

$$\bigcup_{\nu=1}^{\kappa} P_{\lambda_\nu} \uparrow \underset{\kappa = 1, 2, \ldots}{\lambda_\nu \in \Lambda (\nu = 1, 2, \ldots, \kappa)} \bigcup_{\lambda \in \Lambda} P_\lambda.$$

Theorem 13.8. For two sequences of projection operators $P_\nu \uparrow_{\nu=1}^{\infty} P$, and $Q_\nu \uparrow_{\nu=1}^{\infty} Q$, if $P_\nu Q_\mu = Q_\mu P_\nu$ $(\mu, \nu = 1, 2, \ldots)$, then we have

$$P Q_\nu \uparrow_{\nu=1}^{\infty} PQ.$$

Proof. It is obvious by assumption that $P_\nu Q_\nu \uparrow_{\nu=1}^{\infty}$. By virtue of Theorem 13.6, we have further

$$PQ = (\overset{\circ}{\underset{\nu=1}{\cup}} P_\nu)(\overset{\circ}{\underset{\mu=1}{\cup}} Q_\mu) = \overset{\circ}{\underset{\nu,\mu=1}{\cup}} P_\nu Q_\mu = \overset{\circ}{\underset{\nu=1}{\cup}} P_\nu Q_\nu ,$$

because $P_\nu Q_\mu \leqq P_{\nu+\mu} Q_{\nu+\mu}$ for every ν, $\mu = 1, 2, \ldots$.

Theorem 13.9. For projection operators $Q_\lambda \leqq P_\lambda$ $(\lambda \in \Lambda)$, if $P_\lambda P_\gamma = 0$ for $\lambda \neq \gamma$, then we have

$$\bigcup_{\lambda\in\Lambda}(P_\lambda - Q_\lambda) = \bigcup_{\lambda\in\Lambda} P_\lambda - \bigcup_{\lambda\in\Lambda} Q_\lambda .$$

Proof. Since we conclude from assumption by Theorems 12.1 and 12.6 for every λ, $\gamma \in \Lambda$

$$(P_\lambda - Q_\lambda) \cup Q_\lambda = P_\lambda , \quad (P_\lambda - Q_\lambda) Q_\lambda = 0 .$$

we obtain by definition

$$(\bigcup_{\lambda\in\Lambda}(P_\lambda - Q_\lambda)) \cup (\bigcup_{\lambda\in\Lambda} Q_\lambda) = \bigcup_{\lambda\in\Lambda} P_\lambda ,$$

and by Theorem 13.6

$$(\bigcup_{\lambda\in\Lambda}(P_\lambda - Q_\lambda))(\bigcup_{\lambda\in\Lambda} Q_\lambda) = \bigcup_{\lambda,\gamma\in\Lambda}(P_\lambda - Q_\lambda) Q_\gamma = 0 .$$

Thus we conclude by Theorem 12.6

$$\bigcup_{\lambda\in\Lambda}(P_\lambda - Q_\lambda) = \bigcup_{\lambda\in\Lambda} P_\lambda - \bigcup_{\lambda\in\Lambda} Q_\lambda .$$

§14 Rings of Projection Operators

Let R be a Hilbert space. For a collection of operators $\mathfrak{P}$ on R, we shall say that an operator S on R is commutative to $\mathfrak{P}$, if $TS = ST$ for every $T \in \mathfrak{P}$. A collection of operators $\mathfrak{P}$ on R is said to be commutative, if every $T \in \mathfrak{P}$ is commutative to $\mathfrak{P}$. A collection of projection operators $\mathfrak{P}$ is called a ring, if $\mathfrak{P}$ is commutative and

1) P, $Q \in \mathfrak{P}$ implies $P \cup Q \in \mathfrak{P}$,

2) P, $Q \in \mathfrak{P}$ implies $P(1-Q) \in \mathfrak{P}$.

Since $PQ = P(1 - P(1-Q))$, for a ring $\mathfrak{P}$ we have further

3) P, $Q \in \mathfrak{P}$ implies $PQ \in \mathfrak{P}$.

For a ring $\mathfrak{R}$, putting

$$C_{\mathfrak{R}} = \bigcup_{P\in\mathfrak{R}} P ,$$

we obtain a projection operator $C_{\mathfrak{R}}$. This projection operator $C_{\mathfrak{R}}$ is called the cover of $\mathfrak{R}$. A ring $\mathfrak{R}$ is said to be closed, if for every system $P_\lambda \in \mathfrak{R}$ $(\lambda \in \Lambda)$ we have $\bigcup_{\lambda\in\Lambda} P_\lambda \in \mathfrak{R}$, and hence $\bigcap_{\lambda\in\Lambda} P_\lambda \in \mathfrak{R}$, because

$$\bigcap_{\lambda\in\Lambda} P_\lambda = P_{\lambda_0}(1 - \bigcup_{\lambda\neq\lambda_0} P_\lambda) \qquad \text{for } \lambda_0 \in \Lambda .$$

For a closed ring $\mathcal{R}$, we have obviously $C_{\mathcal{R}} \in \mathcal{R}$ and $C_{\mathcal{R}} \geq P$ for every $P \in \mathcal{R}$.

For a collection of operators $\mathcal{P}$ on R, the totality of projection operators which are commutative to $\mathcal{P}$, is called the commutor of $\mathcal{P}$ and denoted by $\overline{\mathcal{P}}$. Concerning the commutor $\overline{\mathcal{P}}$ of $\mathcal{P}$ we have obviously by definition that $1 \in \overline{\mathcal{P}}$ and $1 - P \in \overline{\mathcal{P}}$ for every $P \in \overline{\mathcal{P}}$. If $\mathcal{P}$ is a ring, then we have $\mathcal{P} \subset \overline{\mathcal{P}}$, because every ring $\mathcal{P}$ is commutative by definition.

Let $\mathcal{R}$ be a closed ring of projection operators, $C_{\mathcal{R}}$ the cover of $\mathcal{R}$, and $\overline{\mathcal{R}}$ the commutor of $\mathcal{R}$. Corresponding to every $x \in R$, the projection operator of the closed linear manifold generated by the manifold $\{ Px : P \in \mathcal{R} \}$ is called the projector of x by $\mathcal{R}$, and denoted by $[x]$ or by $[x]_{\mathcal{R}}$, if we need indicate the ring $\mathcal{R}$.

Concerning projectors we have obviously by definition

(1) $\quad [Qx] \leqq C_{\mathcal{R}} Q$ for every $Q \in \overline{\mathcal{R}}$,

(2) $\quad [\alpha x] = [x]$ for every complex number $\alpha \neq 0$.

Since $C_{\mathcal{R}} \in \mathcal{R}$ and $[x]x = [x] C_{\mathcal{R}} x$ by (1) and Theorem 12.1, we obtain by Theorem 11.4

(3) $\quad [x]x = C_{\mathcal{R}} x$.

Thus we have naturally

(4) $\quad [x] = 0$ if and only if $C_{\mathcal{R}} x = 0$.

Denoting by M the closed linear manifold generated by the manifold $\{ Pa : P \in \mathcal{R} \}$, we have by definition $[a] = P_M$. For each $Q \in \mathcal{R}$, denoting by N the closed linear manifold generated by $\{ QPa : P \in \mathcal{R} \}$, we have $QM \subset N$ by Theorem 10.5, and further $N \subset M$, because $QP \in \mathcal{R}$ for every $P \in \mathcal{R}$. Therefore we obtain by Theorem 12.6

$$QP_M = P_M Q, \qquad QM = QN = N.$$

Since $Q \in \mathcal{R}$ may be arbitrary, we obtain hence by definition

(5) $\quad [x] \in \overline{\mathcal{R}}$ for every $x \in R$.

If $Q \in \overline{\mathcal{R}}$, but $QP_M = P_M Q$, then we have $N \subset M$ by Theorem 12.6 and $QM \subset N$ by Theorem 10.5. Thus we conclude

$$QM = QN = N.$$

Since $QP = PQ$ for every $P \in \mathcal{R}$, we obtain $[Qa] = P_N$ by definition,

and further $Q P_M = P_N$ by Theorem 11.6, because $Q P_M$ is a projection operator by Theorem 12.3 and $Q P_M R = QM = N$. Therefore

$$(6) \qquad Q[x] = [Qx] \text{ if } Q \in \overline{\mathfrak{R}}, \quad Q[x] = [x]Q.$$

Since $P[x] = [x]P$ for every $P \in \mathfrak{R}$ by (5), we have especially

$$(6') \qquad P[x] = [Px] \quad \text{for every } P \in \mathfrak{R}.$$

If $[x][y] = 0$, then we have by (1), (3), (5) for every $P \in \mathfrak{R}$

$$(x, Py) = ([x]x, P[y]y) = (x, [x][y]Py) = 0,$$

because $P \leqq C_{\mathfrak{R}}$ for every $P \in \mathfrak{R}$. Conversely if $(x, Py) = 0$ for every $P \in \mathfrak{R}$, then we have

$$(Qx, Py) = (x, QPy) = 0 \quad \text{for every } P, Q \in \mathfrak{R}.$$

Thus, recalling §7(5), we conclude $[x][y] = 0$ by Theorem 12.2. Therefore we have

$$(7) \qquad [x][y] = 0 \text{ if and only if } (x, Py) = 0 \text{ for } P \in \mathfrak{R}.$$

Recalling Theorem 11.5 and the formula §7(5), we see by definition that $(x, Py) = 0$ for every $P \in \mathfrak{R}$ if and only if $[y]x = 0$. We conclude hence from (7)

$$(8) \qquad [x][y] = 0 \text{ if and only if } [y]x = 0.$$

By virtue of (7) and (8), we have $(x - y, Pa) = 0$ for every $P \in \mathfrak{R}$, if and only if $[a](x - y) = 0$. Therefore we obtain

$$(9) \qquad (x, Pa) = (y, Pa) \text{ for } P \in \mathfrak{R} \text{ if and only if } [a]x = [a]y.$$

If $[x]y = C_{\mathfrak{R}}y$, then we have $[x]Py = Py$ for every $P \in \mathfrak{R}$, and hence $[y] \leqq [x]$ by Theorem 11.4. Conversely if $[y] \leqq [x]$, then we have by Theorem 12.1 and (3)

$$[x]y = [x]C_{\mathfrak{R}}y = [x][y]y = [y]y = C_{\mathfrak{R}}y.$$

Thus we obtain

$$(10) \qquad [x] \geqq [y] \text{ if and only if } [x]y = C_{\mathfrak{R}}y.$$

Theorem 14.1. For every $Q \in \overline{\mathfrak{R}}$ we have $C_{\mathfrak{R}}Q = \bigcup_{x \in \mathfrak{f}} [Qx]$.

Proof. For each $x \in R$, we have obviously

$$Q(PQx) = PQx \quad \text{for every } P \in \mathfrak{R}.$$

Thus we see by Theorem 11.4 that $Q \geqq [Qx]$ for every $x \in R$. On the other hand, if $[Qx]y = 0$ for every $x \in R$, then we have by (1) and (3)

$$C_{\mathfrak{R}} Q\, y = [Q\, y]\, Q\, y = [Q\, y]\, y = 0.$$

Thus we conclude by Theorems 11.2 and 13.3 that $\bigcup_{x \in R} [Q\, x] \geqq C_{\mathfrak{R}} Q$. Therefore we obtain $C_{\mathfrak{R}} Q = \bigcup_{x \in R} [Q\, x]$.

Theorem 14.2. If $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty}$ and $\lim_{\nu \to \infty} \| P_\nu a \| = \| a \|$, then we have $P_\nu [a] \uparrow_{\nu=1}^{\infty} [a]$.

Proof. Putting $Q = \bigcup_{\nu=1}^{\infty} P_\nu$, we have $Q \in \mathfrak{R}$ by definition and $Q [a] = \bigcup_{\nu=1}^{\infty} P_\nu [a]$ by Theorem 13.6. Recalling Theorem 13.4, we obtain further by assumption

$$\| Q a \| = \| Q [a] a \| = \lim_{\nu \to \infty} \| P_\nu [a] a \| = \lim_{\nu \to \infty} \| P_\nu a \| = \| a \|.$$

This relation yields $Q a = a$ by Theorem 11.7. Thus we have by (6')

$$[a] = [Q a] = Q [a].$$

Theorem 14.3. For a system $P_\lambda \in \mathfrak{R}$ $(\lambda \in \Lambda)$ and an element $a \in R$, we can find a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ such that

$$\bigcup_{\lambda \in \Lambda} P_\lambda [a] = \bigcup_{\nu=1}^{\infty} P_{\lambda_\nu} [a],$$

$$\bigcap_{\lambda \in \Lambda} P_\lambda [a] = \bigcap_{\nu=1}^{\infty} P_{\lambda_\nu} [a].$$

Proof. We need only prove that for $\mathfrak{R} \ni P_\lambda \uparrow_{\lambda \in \Lambda} P$ we can find a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ such that $P_{\lambda_\nu} [a] \uparrow_{\nu=1}^{\infty} P [a]$. In this case, we have by Theorem 13.4

$$\| P a \| = \| P\, P a \| = \sup_{\lambda \in \Lambda} \| P_\lambda\, P a \|.$$

Thus we can find a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ such that

$$P_{\lambda_\nu} \uparrow_{\nu=1}^{\infty}, \qquad \lim_{\nu \to \infty} \| P_{\lambda_\nu} P a \| = \| P a \|.$$

This relation yields by Theorem 14.2 and (6')

$$P_{\lambda_\nu} [a] = P_{\lambda_\nu} P [a] = P_{\lambda_\nu} [P a] \uparrow_{\nu=1}^{\infty} [P a] = P [a].$$

Theorem 14.4. If $P_{\mu,\nu} [a] \uparrow_{\nu=1}^{\infty} P [a]$ $(\mu = 1, 2, \ldots)$, P, $P_{\mu,\nu} \in \mathfrak{R}$ $(\mu, \nu = 1, 2, \ldots)$, then we can find $\nu_{\mu,\rho}$ $(\mu, \rho = 1, 2, \ldots)$ such that $\bigcap_{\mu=1}^{\infty} P_{\mu,\nu_{\mu,\rho}} [a] \uparrow_{\rho=1}^{\infty} P [a]$.

Proof. By virtue of Theorem 13.5, we obtain by assumption

$$\lim_{\nu \to \infty} \| P a - P_{\mu,\nu} a \| = 0 \qquad \text{for every } \mu = 1, 2, \ldots .$$

Thus we can find $\nu_{\mu,\rho}$ $(\mu, \rho = 1, 2, \ldots)$ such that $\nu_{\mu,\rho} < \nu_{\mu,\rho+1}$ and

$$\| P a - P_{\mu,\nu_{\mu,\rho}} a \| \leqq \frac{1}{2^{\mu+\rho}} \qquad (\mu, \rho = 1, 2, \ldots).$$

Then we have by Theorem 13.7 and the formula §13(4)

$$\| P a - \bigcap_{\mu=1}^{\infty} P_{\mu,\nu_{\mu,\rho}} a \| = \| \bigcup_{\mu=1}^{\infty} (1 - P_{\mu,\nu_{\mu,\rho}}) P a \|$$

$$\leqq \sum_{\mu=1}^{\infty} \| P a - P_{\mu,\nu_{\mu,\rho}} a \| \leqq \frac{1}{2^{\rho}}$$

for every $\rho = 1, 2, \ldots$, and hence $\lim_{\rho\to\infty} \bigcap_{\mu=1}^{\infty} P_{\mu,\nu_{\mu,\rho}} a = a$. Thus we have by Theorem 14.2 $\bigcap_{\mu=1}^{\infty} P_{\mu,\nu_{\mu,\rho}} [a] \uparrow_{\rho=1}^{\infty} P[a]$.

For an arbitrary projection operator Q we define the <u>cover</u> C_Q of Q by $\mathfrak{R}$ to mean

$$C_Q = C_{\mathfrak{R}} - \bigcup_{QP=0,\, P\in\mathfrak{R}} P = C_{\mathfrak{R}} \bigwedge_{Q\leqq 1-P,\, P\in\mathfrak{R}} (1 - P).$$

With this definition, it is evident that $C_Q \in \mathfrak{R}$ for every projection operator Q, and we have obviously

(11) $\quad C_P = P \quad$ for every $P \in \mathfrak{R}$,

(12) $\quad Q_1 \leqq Q_2$ implies $C_{Q_1} \leqq C_{Q_2}$.

Recalling Theorem 13.6, we have

$$Q \bigcup_{QP=0,\, P\in\mathfrak{R}} P = 0,$$

and hence

(13) $\quad Q \frown C_{\mathfrak{R}} \leqq C_Q$,

(14) $\quad C_Q P = 0$, $P \in \mathfrak{R}$, if and only if $QP = 0$.

For a system of projection operators Q_λ ($\lambda \in \Lambda$) we see by (14) that we have

$$C_{\bigcup_{\lambda\in\Lambda} Q_\lambda} P = 0, \quad P \in \mathfrak{R},$$

if and only if $\bigcup_{\lambda\in\Lambda} Q_\lambda P = 0$, and hence by Theorem 13.6, if and only if $Q_\lambda P = 0$ for every $\lambda \in \Lambda$. Therefore we have by (14)

(15) $\quad C_{\bigcup_{\lambda\in\Lambda} Q_\lambda} = \bigcup_{\lambda\in\Lambda} C_{Q_\lambda}$.

For $P_0 \in \mathfrak{R}$, $Q \in \overline{\mathfrak{R}}$, we have by (14) $C_{P_0 Q} P = 0$, $P \in \mathfrak{R}$, if and only if $P P_0 Q = 0$, and hence if and only if $C_Q P_0 P = 0$. Thus we conclude $C_{P_0 Q}(C_{\mathfrak{R}} - C_Q P_0) = C_Q P_0 (C_{\mathfrak{R}} - C_{P_0 Q}) = 0$, and hence $C_{P_0 Q} = P_0 C_Q$. Therefore we obtain

(16) $\quad C_{PQ} = P C_Q \quad$ for $P \in \mathfrak{R}$, $Q \in \mathfrak{R}$.

Since $[x] \in \overline{\mathfrak{R}}$ by (5), putting $Q = [x]$ in (16), we obtain by (6')

(16') $\quad C_{[Px]} = C_{P[x]} = P C_{[x]} \quad$ for $P \in \mathfrak{R}$.

Since $[x] \leqq C_{[x]}$ by (13), we have by (3)

$$C_{[x]}x = C_{[x]}C_{\mathfrak{R}}x = C_{[x]}[x]x = [x]x = C_{\mathfrak{R}}x,$$

that is,

$$C_{[x]}x = [x]x = C_{\mathfrak{R}}x. \tag{17}$$

If $Px = C_{\mathfrak{R}}x$, $P \in \mathfrak{R}$, then we have by (16')

$$PC_{[x]} = C_{[Px]} = C_{[C_{\mathfrak{R}}x]} = C_{\mathfrak{R}}C_{[x]} = C_{[x]}.$$

Conversely if $PC_{[x]} = C_{[x]}$, then we have by (17)

$$C_{\mathfrak{R}}x = C_{[x]}x = PC_{[x]}x = PC_{\mathfrak{R}}x = Px.$$

Therefore we obtain by Theorem 12.1

$$P \geqq C_{[x]},\ P \in \mathfrak{R}, \text{ if and only if } Px = C_{\mathfrak{R}}x. \tag{18}$$

If $PC_{[x]} = 0$, $P \in \mathfrak{R}$, then we have by (17)

$$Px = PC_{\mathfrak{R}}x = PC_{[x]}x = 0.$$

Conversely if $Px = 0$, $P \in \mathfrak{R}$, then we have by (16') $PC_{[x]} = 0$. Thus we have

$$PC_{[x]} = 0,\ P \in \mathfrak{R}, \text{ if and only if } Px = 0. \tag{19}$$

From the definition of covers we conclude immediately

$$C_{\mathfrak{A}} = \bigcap_{\mathfrak{A} \leqq P \in \mathfrak{R}} P \qquad \text{for } \mathfrak{A} \leqq C_{\mathfrak{R}}. \tag{20}$$

Theorem 14.5. If $C_{[a_\nu]}C_{[a_\mu]} = 0$ for $\nu \neq \mu$ and $\sum_{\nu=1}^{\infty} a_\nu$ is convergent, then $[\sum_{\nu=1}^{\infty} a_\nu] = \bigcup_{\nu=1}^{\infty} [a_\nu]$.

Proof. Since $C_{[a_\mu]}C_{[a_\nu]} = 0$ for $\mu \neq \nu$ by assumption, we have by (19) $C_{[a_\mu]}a_\nu = 0$ for $\mu \neq \nu$, and hence by (17)

$$C_{[a_\mu]} \sum_{\nu=1}^{\infty} a_\nu = C_{[a_\mu]}a_\mu = C_{\mathfrak{R}}a_\mu$$

for every $\mu = 1, 2, \ldots$. Thus we obtain by (6')

$$C_{[a_\mu]}[\sum_{\nu=1}^{\infty} a_\nu] = [C_{\mathfrak{R}}a_\mu] = C_{\mathfrak{R}}[a_\mu] = [a_\mu]$$

for every $\mu = 1, 2, \ldots$. This relation yields by Theorem 13.6 and the formula (6')

$$\bigcup_{\mu=1}^{\infty}[a_\mu] = \bigcup_{\mu=1}^{\infty} C_{[a_\mu]}[\sum_{\nu=1}^{\infty} a_\nu] = [\bigcup_{\mu=1}^{\infty} C_{[a_\mu]} \sum_{\nu=1}^{\infty} a_\nu]$$
$$= [\sum_{\nu=1}^{\infty} \bigcup_{\mu=1}^{\infty} C_{[a_\mu]}a_\nu] = [\sum_{\nu=1}^{\infty} a_\nu],$$

because we have by (17)

$$\bigcup_{\mu=1}^{\infty} C_{[a_\mu]}a_\nu = \bigcup_{\mu=1}^{\infty} C_{[a_\mu]}C_{\mathfrak{R}}a_\nu = \bigcup_{\mu=1}^{\infty} C_{[a_\mu]}C_{[a_\nu]}a_\nu$$
$$= C_{[a_\nu]}a_\nu = C_{\mathfrak{R}}a_\nu.$$

Theorem 14.6. For any sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ we can find $a \in R$ such that $C_{[a]} = \bigcup_{\nu=1}^{\infty} C_{[a_\nu]}$.

Proof. Putting $b_1 = a_1$, $b_\nu = (C_{\mathfrak{R}} - \bigcup_{\mu=1}^{\nu-1} C_{[a_\mu]})\, a_\nu$ for $\nu = 2,3,\ldots$, we have by (16') for every $\nu = 2, 3, \ldots$

$$C_{[b_\nu]} = (C_{\mathfrak{R}} - \bigcup_{\mu=1}^{\nu-1} C_{[a_\mu]})\, C_{[a_\nu]}.$$

Thus we have $C_{[b_\nu]} C_{[b_\mu]} = 0$ for $\nu \neq \mu$ and further

$$\bigcup_{\nu=1}^{\infty} C_{[b_\nu]} = \bigcup_{\nu=1}^{\infty} C_{[a_\nu]},$$

because we see by Theorem 12.6 for every $\nu = 2, 3, \ldots$

$$C_{[b_\nu]} = \bigcup_{\mu=1}^{\nu} C_{[a_\mu]} - \bigcup_{\mu=1}^{\nu-1} C_{[a_\mu]}.$$

Now we can find $\alpha_\nu > 0$ $(\nu = 1, 2, \ldots)$ such that $\sum_{\nu=1}^{\infty} \|\alpha_\nu b_\nu\|^2 < +\infty$ and then we can put by Theorem 7.11

$$b = \sum_{\nu=1}^{\infty} \alpha_\nu b_\nu.$$

Since $[\alpha_\nu b_\nu] = [b_\nu]$ by (2), we have then by Theorem 14.5

$$[b] = \bigcup_{\nu=1}^{\infty} [b_\nu],$$

and hence by (15) $C_{[b]} = \bigcup_{\nu=1}^{\infty} C_{[b_\nu]} = \bigcup_{\nu=1}^{\infty} C_{[a_\nu]}$.

Theorem 14.7. For a projection operator $P \in \mathfrak{R}$, we can find an element $a \in R$ such that $P = C_{[a]}$ if and only if for any system $P_\lambda \in \mathfrak{R}$ $(\lambda \in \Lambda)$ we can find a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ such that

$$P \bigcup_{\lambda \in \Lambda} P_\lambda = P \bigcup_{\nu=1}^{\infty} P_{\lambda_\nu}.$$

Proof. If $P = C_{[a]}$ for some $a \in R$, then for any system $P_\lambda \in R$ $(\lambda \in \Lambda)$ we can find by Theorem 14.3 a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ such that we have

$$\bigcup_{\lambda \in \Lambda} P_\lambda [a] = \bigcup_{\nu=1}^{\infty} P_{\lambda_\nu} [a],$$

and hence by (16')

$$\bigcup_{\lambda \in \Lambda} P_\lambda C_{[a]} = \bigcup_{\nu=1}^{\infty} P_{\lambda_\nu} C_{[a]}.$$

Conversely, if for any system $P_\lambda \in \mathfrak{R}$ $(\lambda \in \Lambda)$ we can find a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ for which

$$P \bigcup_{\lambda \in \Lambda} P_\lambda = P \bigcup_{\nu=1}^{\infty} P_{\lambda_\nu},$$

then, since $P = \bigcup_{x \in R} [Px]$ by Theorem 14.1, we have by (11) and (15)

$$P = \bigcup_{x \in R} C_{[Px]},$$

and hence we can find a sequence $x_\nu \in R$ $(\nu = 1, 2, \ldots)$ for which

$$P = \bigcup_{\nu=1}^{\infty} C_{[Px_\nu]}.$$

Then there exists by Theorem 14.6 $a \in R$ such that $P = C_{[a]}$.

§15 Proper Spaces

Let $\mathfrak{R}$ be a closed ring of projection operators and $C_{\mathfrak{R}}$ the cover of $\mathfrak{R}$. A subset $\mathfrak{P}$ of $\mathfrak{R}$ is called an ideal, if

1) $0 \in \mathfrak{P}$,

2) $P \in \mathfrak{P}$, $P \leqq Q \in \mathfrak{R}$ implies $Q \in \mathfrak{P}$,

3) $P, Q \in \mathfrak{P}$ implies $PQ \in \mathfrak{P}$.

An ideal $\mathfrak{P}$ is said to be maximal, if there exists no other ideal including $\mathfrak{P}$.

From the postulates 1) and 3) we conclude immediately that every ideal $\mathfrak{P}$ satisfies the condition

(F) $\bigcap_{\nu=1}^{\varkappa} P_\nu \neq 0$ for finite $P_\nu \in \mathfrak{P}$ $(\nu = 1, 2, \ldots, \varkappa)$.

Theorem 15.1. If a subset $\mathfrak{P}_0$ of $\mathfrak{R}$ satisfies the condition (F), then there exists a maximal ideal including $\mathfrak{P}_0$.

Proof. By virtue of Maximal Theorem, we see easily that there exists a maximal subset $\mathfrak{P}$ of $\mathfrak{R}$ which includes $\mathfrak{P}_0$ and satisfies the condition (F). Such $\mathfrak{P}$ is a maximal ideal. In fact, we have obviously $0 \in \mathfrak{P}$. If $P \in \mathfrak{P}$ and $P \leqq Q \in \mathfrak{R}$, then we have $Q \in \mathfrak{P}$. Because, if $Q \bar{\in} \mathfrak{P}$, then $\{\mathfrak{P}, Q\}$ satisfies obviously the condition (F), contradicting that $\mathfrak{P}$ is a maximal subset satisfying (F). Finally, if $P, Q \in \mathfrak{P}$, then we have $PQ \in \mathfrak{P}$. Because, if $PQ \bar{\in} \mathfrak{P}$, then $\{\mathfrak{P}, PQ\}$ satisfies (F), contradicting the maximality of $\mathfrak{P}$. Therefore $\mathfrak{P}$ is an ideal by definition. Furthermore $\mathfrak{P}$ is a maximal ideal, because every ideal satisfies the condition (F).

Theorem 15.2. In order that an ideal $\mathfrak{P}$ be maximal, it is necessary and sufficient that for $\mathfrak{R} \ni Q \bar{\in} \mathfrak{P}$ we can find $P \in \mathfrak{P}$ such that $PQ = 0$.

Proof. Let $\mathfrak{P}$ be an ideal. If there exists $Q \in \mathfrak{R}$ such that

$Q \bar{\in} \mathfrak{P}$ and $PQ \neq 0$ for every $P \in \mathfrak{P}$, then $\{\mathfrak{P}, Q\}$ satisfies obviously the condition (F), and hence there exists by Theorem 15.1 a maximal ideal including $\{\mathfrak{P}, Q\}$. Therefore $\mathfrak{P}$ is not maximal.

Conversely, if $\mathfrak{P}$ is not maximal, then there exists by definition an ideal $\mathfrak{P}_0$ including $\mathfrak{P}$ but different from $\mathfrak{P}$. For such $\mathfrak{P}_0$ we can find $Q \in \mathfrak{P}_0$ such that $Q \bar{\in} \mathfrak{P}$, but we have obviously by definition $PQ \neq 0$ for every $P \in \mathfrak{P}$.

Theorem 15.3. Let $\mathfrak{P}$ be a maximal ideal. If

$$\bigcup_{\nu=1}^{\varkappa} P_\nu \in \mathfrak{P}, \qquad P_\nu \in \mathfrak{R} \quad (\nu = 1, 2, \ldots, \varkappa),$$

then there exists ν for which $P_\nu \in \mathfrak{P}$.

Proof. If $P_\nu \bar{\in} \mathfrak{P}$ for every $\nu = 1, 2, \ldots, \varkappa$, then there is by Theorem 15.2 $Q_\nu \in \mathfrak{P}$ $(\nu = 1, 2, \ldots, \varkappa)$ such that $P_\nu Q_\nu = 0$ for every $\nu = 1, 2, \ldots, \varkappa$. For such $Q_\nu \in \mathfrak{P}$ $(\nu = 1, 2, \ldots, \varkappa)$, putting

$$Q = \bigcap_{\nu=1}^{\varkappa} Q_\nu,$$

we have $Q \in \mathfrak{P}$ by the postulate 3), and hence $P_\nu Q = 0$ for every $\nu = 1, 2, \ldots, \varkappa$. Consequently we have by Theorem 13.6

$$(\bigcup_{\nu=1}^{\varkappa} P_\nu) Q = 0,$$

and hence $\bigcup_{\nu=1}^{\varkappa} P_\nu \bar{\in} \mathfrak{P}$ by the postulates 1) and 3).

Considering every maximal ideal as a point, we obtain a space $\mathfrak{C}_{\mathfrak{R}}$ which is composed of all maximal ideals in $\mathfrak{R}$. Corresponding to every $P \in \mathfrak{R}$, we define U_P to mean the set of all maximal ideals which contain P, and naturally $U_0 = 0$, $U_{e_{\mathfrak{R}}} = \mathfrak{C}_{\mathfrak{R}}$. With this definition we have obviously

(1) $\mathfrak{P} \in U_P$ if and only if $P \in \mathfrak{P}$.

(2) $U_P U_Q = U_{PQ}$.

Because, if $\mathfrak{P} \in U_P U_Q$, then we have $P, Q \in \mathfrak{P}$ by (1), and hence we obtain $PQ \in \mathfrak{P}$ by 3), that is, $\mathfrak{P} \in U_{PQ}$ by (1). Conversely, if $\mathfrak{P} \in U_{PQ}$, then we have $PQ \in \mathfrak{P}$, and hence $P, Q \in \mathfrak{P}$ by 2), that is, $\mathfrak{P} \in U_P U_Q$ by (1).

By virtue of Theorem 15.3 and the postulate 2), we have $P \cup Q \in \mathfrak{P}$ if and only if $P \in \mathfrak{P}$ or $Q \in \mathfrak{P}$. Therefore we obtain by (1)

(3) $U_P + U_Q = U_{P \cup Q}$.

$U_P \supset U_Q$ is obviously equivalent to $U_P U_Q = U_Q$, and $P \geqq Q$ is equivalent to $PQ = Q$ by Theorem 12.1. Furthermore $U_{PQ} = U_Q$ implies $PQ = Q$. Because, if $Q - PQ \neq 0$, then, since $Q - PQ \in \mathfrak{R}$, there exists by Theorem 15.1 a maximal ideal $\mathfrak{P} \ni Q - PQ$. For such $\mathfrak{P}$, since $Q - PQ \leqq Q$, $QP(Q - PQ) = 0$, we have $Q \in \mathfrak{P}$ but $QP \bar{\in} \mathfrak{P}$, and hence $\mathfrak{P} \in U_Q$ but $\mathfrak{P} \bar{\in} U_{PQ}$, contradicting the assumption $U_{PQ} = U_Q$. Therefore we obtain by (2)

(4) $\quad U_P \supset U_Q$ if and only if $P \geqq Q$.

By virtue of this formula (4) we see that $U_P = 0$ implies $P = 0$. Therefore we obtain by (2)

(5) $\quad U_P U_Q = 0$ if and only if $PQ = 0$.

Since we have by Theorem 12.6

$$P \smile (C_{\mathfrak{R}} - P) = C_{\mathfrak{R}} \qquad \text{for } P \in \mathfrak{R},$$

we obtain by (3) and (5)

$$\mathfrak{C}_{\mathfrak{R}} = U_P \dotplus U_{C_{\mathfrak{R}} - P}, \qquad U_P U_{C_{\mathfrak{R}} - P} = 0.$$

Thus we conclude

(6) $\quad U_P' = U_{C_{\mathfrak{R}} - P}$.

We can introduce a topology into this space $\mathfrak{C}_{\mathfrak{R}}$ such that the totality of U_P $(P \in \mathfrak{R})$ constitutes a neighbourhood system, because U_P $(P \in \mathfrak{R})$ satisfies the neighbourhood conditions by (1) and (2). This topological space $\mathfrak{C}_{\mathfrak{R}}$ is called the proper space of a ring $\mathfrak{R}$, and U_P a neighbourhood by $P \in \mathfrak{R}$.

Theorem 15.4. *The proper space $\mathfrak{C}_{\mathfrak{R}}$ of a ring $\mathfrak{R}$ is a compact Hausdorff space and every neighbourhood U_P $(P \in \mathfrak{R})$ is open and closed.*

Proof. By virtue of the formula (6), it is obvious that every neighbourhood U_P $(P \in \mathfrak{R})$ is open and closed.

For two different maximal ideals $\mathfrak{P}_1$ and $\mathfrak{P}_2$, there exists obviously $P_1 \in \mathfrak{P}_1$ such that $P_1 \bar{\in} \mathfrak{P}_2$. For such P_1 we can find by Theorem 15.2 $P_2 \in \mathfrak{P}_2$ such that $P_1 P_2 = 0$. Then we have by the formulas (1) and (5)

$$\mathfrak{P}_1 \in U_{P_1}, \qquad \mathfrak{P}_2 \in U_{P_2}, \qquad U_{P_1} U_{P_2} = 0.$$

Therefore $\mathfrak{C}_{\mathfrak{R}}$ is a Hausdorff space.

If $\mathfrak{C}_{\mathfrak{R}} = \sum_{\lambda \in \Lambda} U_{P_\lambda}$, then we have by the formula (6)

$$\prod_{\lambda \in \Lambda} U_{C_{\mathfrak{R}} - P_\lambda} = \prod_{\lambda \in \Lambda} U_{P_\lambda}' = 0.$$

If $\prod_{\nu=1}^{\kappa} U_{C_{\mathfrak{R}} - P_{\lambda_\nu}} \neq 0$ for every finite number of elements $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \kappa$), then we see easily by (2) that the system $C_{\mathfrak{R}} - P_\lambda$ ($\lambda \in \Lambda$) satisfies the condition (F), and hence there exists by Theorem 15.1 a maximal ideal $\mathfrak{F}$ such that $C_{\mathfrak{R}} - P_\lambda \in \mathfrak{F}$ for every $\lambda \in \Lambda$, that is,

$$\mathfrak{F} \in \prod_{\lambda \in \Lambda} U_{C_{\mathfrak{R}} - P_\lambda},$$

contradicting $\prod_{\lambda \in \Lambda} U_{C_{\mathfrak{R}} - P_\lambda} = 0$. Therefore there exists a finite number of elements $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \kappa$) such that

$$\prod_{\nu=1}^{\kappa} U_{C_{\mathfrak{R}} - P_{\lambda_\nu}} = 0,$$

and hence $\mathfrak{C}_{\mathfrak{R}} = \sum_{\nu=1}^{\kappa} U_{P_{\lambda_\nu}}$. Thus $\mathfrak{C}_{\mathfrak{R}}$ is compact.

Theorem 15.5. *In the proper space $\mathfrak{C}_{\mathfrak{R}}$, if a closed set A is included in an open set B, then there exists $P \in \mathfrak{R}$ such that*

$$A \subset U_P \subset B.$$

Proof. Since B is open by assumption, we can find by definition a system $P_\lambda \in \mathfrak{R}$ ($\lambda \in \Lambda$) for which $B = \sum_{\lambda \in \Lambda} U_{P_\lambda}$. Since A is closed by assumption, A is compact by Theorem 15.4, and further we have $A \subset \sum_{\lambda \in \Lambda} U_{P_\lambda}$ by assumption. Thus we can find a finite number of elements $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \ldots, \kappa$) such that $A \subset \sum_{\nu=1}^{\kappa} U_{P_{\lambda_\nu}}$. Then, putting $P = \bigcup_{\nu=1}^{\kappa} P_{\lambda_\nu}$, we obtain $A \subset U_P \subset B$ by (3).

As an immediate consequence from Theorem 15.5, we have

Theorem 15.6. *For a closed open set A, we can find uniquely $P \in \mathfrak{R}$ for which $A = U_P$.*

Theorem 15.7. *We have $U_P = (\sum_{\lambda \in \Lambda} U_{P_\lambda})^-$, if and only if*

$$P = \bigcup_{\lambda \in \Lambda} P_\lambda.$$

Proof. If $P = \bigcup_{\lambda \in \Lambda} P_\lambda$, then we have $U_P \supset \sum_{\lambda \in \Lambda} U_{P_\lambda}$ by (4). Since U_P is closed by Theorem 15.4, we obtain hence

$$U_P \supset (\sum_{\lambda \in \Lambda} U_{P_\lambda})^-.$$

Furthermore, if $U_Q \subset U_P - (\sum_{\lambda \in \Lambda} U_{P_\lambda})^-$, then we have by (4) and (5)

$$Q \leqq P, \qquad Q P_\lambda = 0 \quad \text{for every } \lambda \in \Lambda,$$

and consequently by Theorems 12.1 and 13.6

$$0 = Q P = \bigcup_{\lambda \in \Lambda} Q P_\lambda = 0 .$$

Therefore we obtain $U_P = (\sum_{\lambda \in \Lambda} U_{P_\lambda})^-$.

Conversely, if $U_P = (\sum_{\lambda \in \Lambda} U_{P_\lambda})^-$, then, putting $Q = \bigcup_{\lambda \in \Lambda} P_\lambda$, we have

$$U_Q = (\sum_{\lambda \in \Lambda} U_{P_\lambda})^- = U_P ,$$

as proved just above. This relation yields $P = Q$ by (4).

Theorem 15.8. We have $U_P = (\prod_{\lambda \in \Lambda} U_{P_\lambda})^\circ$, if and only if

$$P = \bigcap_{\lambda \in \Lambda} P_\lambda .$$

Proof. $U_P = (\prod_{\lambda \in \Lambda} U_{P_\lambda})^\circ$ is by (6) equivalent to

$$U_{C_{\mathfrak{R}} - P} = (\sum_{\lambda \in \Lambda} U_{C_{\mathfrak{R}} - P_\lambda})^- ,$$

and $P = \bigcap_{\lambda \in \Lambda} P_\lambda$ is by §12(4) equivalent to

$$C_{\mathfrak{R}} - P = \bigcup_{\lambda \in \Lambda} (C_{\mathfrak{R}} - P_\lambda) .$$

Thus we conclude our assertion from Theorem 15.7.

Theorem 15.9. For every open set A, its closure A^- also is open. For every closed set A, its opener A° also is closed.

Proof. If A is open, then we can find by definition a system $P_\lambda \in \mathfrak{R}$ $(\lambda \in \Lambda)$ such that $A = \sum_{\lambda \in \Lambda} U_{P_\lambda}$. Putting $P = \bigcup_{\lambda \in \Lambda} P_\lambda$, we have then by Theorem 15.7

$$U_P = (\sum_{\lambda \in \Lambda} U_{P_\lambda})^- = A^- .$$

Thus A^- is open by Theorem 15.4.

If A is closed, then its complement A' is open, and hence A'^- also is open, as proved just above. Since $A'^- = A^{\circ\prime}$, we see that A° is closed.

Theorem 15.10. For every sequence of nowhere dense sets A_ν $(\nu = 1, 2, \ldots)$, its union $\sum_{\nu=1}^{\infty} A_\nu$ also is nowhere dense.

Proof. Since $A_\nu^{-\circ} = 0$ by assumption, we have for every $x \in R$

$$(U_{C_{(x)}} A_\nu^{-\prime})^- = U_{C_{(x)}} .$$

Since $U_{C_{(x)}} A_\nu^{-\prime}$ is open, we can find $P_\lambda \in \mathfrak{R}$ $(\lambda \in \Lambda)$ such that

$$U_{C_{(x)}} A_\nu^{-\prime} = \sum_{\lambda \in \Lambda} U_{P_\lambda} ,$$

and hence by Theorem 15.7

$$\mathcal{C}_{[x]} = \bigcup_{\lambda \in \Lambda} P_\lambda .$$

Recalling Theorem 14.3, we can find therefore $P_{\nu,\mu} \in \mathcal{R}$ ($\nu, \mu = 1, 2, \ldots$) such that

$$\mathcal{U}_{\mathcal{C}_{[x]}} A_\nu^{-\prime} > \sum_{\mu=1}^{\infty} \mathcal{U}_{P_{\nu,\mu}} \quad (\nu = 1, 2, \ldots),$$

$$P_{\nu,\mu}[x] \uparrow_{\mu=1}^{\infty} \mathcal{C}_{[x]}[x] = [x].$$

Furthermore we can find by Theorem 14.4 $\mu_{\nu,\rho}$ ($\nu, \rho = 1, 2, \ldots$) such that

$$\bigcap_{\nu=1}^{\infty} P_{\nu,\mu_{\nu,\rho}}[x] \uparrow_{\rho=1}^{\infty} [x].$$

Then, putting

$$Q_\rho = \bigcap_{\nu=1}^{\infty} P_{\nu,\mu_{\nu,\rho}} \mathcal{C}_{[x]} \qquad (\rho = 1, 2, \ldots),$$

we obtain by the formulas (15) and (16') in §14

$$Q_\rho \uparrow_{\rho=1}^{\infty} \mathcal{C}_{[x]},$$

and hence by Theorem 15.7

$$\mathcal{U}_{\mathcal{C}_{[x]}} = (\sum_{\rho=1}^{\infty} \mathcal{U}_{Q_\rho})^- .$$

On the other hand we have obviously by (4) for every $\rho = 1, 2, \ldots$

$$\mathcal{U}_{Q_\rho} \subset \prod_{\nu=1}^{\infty} \mathcal{U}_{\mathcal{C}_{[x]}} A_\nu^{-\prime} = \mathcal{U}_{\mathcal{C}_{[x]}} (\sum_{\nu=1}^{\infty} A_\nu^-)' .$$

This relation yields

$$\sum_{\rho=1}^{\infty} \mathcal{U}_{Q_\rho} \subset (\mathcal{U}_{\mathcal{C}_{[x]}} (\sum_{\nu=1}^{\infty} A_\nu^-)')^\circ = \mathcal{U}_{\mathcal{C}_{[x]}} (\sum_{\nu=1}^{\infty} A_\nu^-)^{-\prime} .$$

Thus we obtain

$$\mathcal{U}_{\mathcal{C}_{[x]}} = (\mathcal{U}_{\mathcal{C}_{[x]}} (\sum_{\nu=1}^{\infty} A_\nu^-)^{-\prime})^- = \mathcal{U}_{\mathcal{C}_{[x]}} (\sum_{\nu=1}^{\infty} A_\nu^-)^{-\circ\prime}.$$

Therefore we have

$$\mathcal{U}_{\mathcal{C}_{[x]}} (\sum_{\nu=1}^{\infty} A_\nu^-)^{-\circ} = 0 \qquad \text{for every } x \in R .$$

Since $\mathcal{C}_{\mathcal{R}} = \bigcup_{x \in R} [x]$ by Theorem 14.1, we have $\mathcal{C}_{\mathcal{R}} = \bigcup_{x \in R} \mathcal{C}_{[x]}$ by (15) in §14, and hence we obtain by Theorem 15.7

$$\mathcal{C}_{\mathcal{R}} = (\sum_{x \in R} \mathcal{U}_{\mathcal{C}_{[x]}})^- .$$

Therefore we have

$$(\sum_{\nu=1}^{\infty} A_\nu^-)^{-\circ} = (\sum_{x \in R} \mathcal{U}_{\mathcal{C}_{[x]}})^- (\sum_{\nu=1}^{\infty} A_\nu^-)^{-\circ}$$

$$\subset (\sum_{x \in R} \mathcal{U}_{\mathcal{C}_{[x]}} (\sum_{\nu=1}^{\infty} A_\nu^-)^{-\circ})^- = 0 .$$

Thus $\sum_{\nu=1}^{\infty} A_\nu$ is nowhere dense.

From this Theorem 15.10 we conclude immediately

Theorem 15.11. For any sequence of open dense sets A_ν ($\nu = 1, 2, \ldots$), $(\prod_{\nu=1}^{\infty} A_\nu)^\circ$ also is dense.

Because, for a point set A, its opener A° is dense, if and only if its complement A' is nowhere dense, and we have

$$(\prod_{\nu=1}^{\infty} A_\nu)' = \sum_{\nu=1}^{\infty} A_\nu' .$$

§16 Integration

Let $\mathfrak{R}$ be a closed ring of projection operators. A finite function $\mu(P)$ ($P \in \mathfrak{R}$) on $\mathfrak{R}$ is said to be additive, if $PQ = 0$ implies

$$\mu(P+Q) = \mu(P) + \mu(Q).$$

For an additive function μ on $\mathfrak{R}$ we have obviously by definition

$$\mu(0) = 0,$$
$$\mu(P-Q) = \mu(P) - \mu(Q) \qquad \text{for } P \geqq Q.$$

A function μ on $\mathfrak{R}$ is said to be totally additive, if μ is additive and $\mathfrak{R} \ni P_\nu \downarrow_{\nu=1}^{\infty} 0$ implies

$$\lim_{\nu \to \infty} \mu(P_\nu) = 0.$$

For a totally additive function μ on $\mathfrak{R}$, if $P_\nu \uparrow_{\nu=1}^{\infty} P$ or $P_\nu \downarrow_{\nu=1}^{\infty} P$, then we have

$$\lim_{\nu \to \infty} \mu(P_\nu) = \mu(P).$$

Because, if $P_\nu \uparrow_{\nu=1}^{\infty} P$ or $P_\nu \downarrow_{\nu=1}^{\infty} P$, then we have $P - P_\nu \downarrow_{\nu=1}^{\infty} 0$ or $P_\nu - P \downarrow_{\nu=1}^{\infty} 0$ respectively, and hence $\lim_{\nu \to \infty} \mu(P_\nu) = \mu(P)$.

Theorem 16.1. If a real valued function μ on $\mathfrak{R}$ is totally additive, then we can find $P_0 \in \mathfrak{R}$ such that

$$\left.\begin{aligned} \mu(PP_0) &\geqq 0 \\ \mu((1-P)P_0) &\leqq 0 \end{aligned}\right\} \quad \text{for every } P \in \mathfrak{R}.$$

Proof. We shall prove firstly $\sup_{P \in \mathfrak{R}} |\mu(P)| < +\infty$. If $\sup_{P \in \mathfrak{R}} |\mu(P)| = +\infty$, then we can find $Q \in \mathfrak{R}$ such that

$$|\mu(Q)| \geqq 1 + |\mu(C_{\mathfrak{R}})| .$$

For such Q, we have obviously

$$|\mu(C_{\mathfrak{R}} - Q)| \geqq |\mu(Q)| - |\mu(C_{\mathfrak{R}})| \geqq 1,$$

$$\sup_{P\in\mathfrak{R}} |\mu(PQ)| = +\infty \quad \text{or} \quad \sup_{P\in\mathfrak{R}} |\mu((1-Q)P)| = +\infty,$$

because

$$|\mu(P)| \leqq |\mu(PQ)| + |\mu((1-Q)P)| .$$

If we put $P_1 = Q$ or $P_1 = \mathfrak{C}_{\mathfrak{R}} - Q$ corresponding to

$$\sup_{P\in\mathfrak{R}} |\mu(QP)| = +\infty \quad \text{or} \quad \sup_{P\in\mathfrak{R}} |\mu((1-Q)P)| = +\infty$$

respectively, then we have

$$|\mu(P_1)| \geqq 1, \qquad \sup_{P\in\mathfrak{R}} |\mu((1-P_1)P)| = +\infty.$$

We can find likewise consecutively $P_\nu \in \mathfrak{R}$ $(\nu = 2, 3, \ldots)$ such that

$$P_\nu P_\mu = 0 \qquad \text{for } \nu \neq \mu,$$

$$|\mu(P_\nu)| \geqq \nu + |\mu(\bigcup_{\rho=1}^{\nu-1} P_\rho)|,$$

$$\sup_{P\in\mathfrak{R}} |\mu((1-\bigcup_{\rho=1}^{\nu} P_\rho)P)| = +\infty .$$

For such $P_\nu \in \mathfrak{R}$ $(\nu = 1, 2, \ldots)$, putting $P_0 = \bigcup_{\nu=1}^{\infty} P_\nu$, we have

$$\bigcup_{\rho=1}^{\nu} P_\rho \uparrow_{\nu=1}^{\infty} P_0,$$

$$|\mu(\bigcup_{\rho=1}^{\nu} P_\rho)| \geqq |\mu(P_\nu)| - |\mu(\bigcup_{\rho=1}^{\nu-1} P_\rho)| \geqq \nu,$$

contradicting $\lim_{\nu\to\infty} \mu(\bigcup_{\rho=1}^{\nu} P_\rho) = \mu(P_0)$.

Therefore, putting $\alpha = \sup_{P\in\mathfrak{R}} \mu(P)$, we have $0 \leqq \alpha < +\infty$.

If for some $P, Q \in \mathfrak{R}$ and positive numbers ε, δ we have

$$\mu(P) > \alpha - \varepsilon, \qquad \mu(Q) > \alpha - \delta,$$

then, since $\mu(PQ) \leqq \alpha$, we obtain by Theorem 12.6

$$\mu(P \vee Q) = \mu(P + (1-P)Q)$$
$$= \mu(P) + \mu(Q) - \mu(PQ) \geqq \alpha - (\varepsilon + \delta).$$

There exists obviously a sequence $P_\nu \in \mathfrak{R}$ $(\nu = 1, 2, \ldots)$ such that

$$\mu(P_\nu) > \alpha - \frac{1}{2^\nu} \qquad (\nu = 1, 2, \ldots).$$

For such $P_\nu \in \mathfrak{R}$ $(\nu = 1, 2, \ldots)$, putting

$$Q_\rho = \bigcup_{\nu=\rho}^{\infty} P_\nu \qquad (\rho = 1, 2, \ldots),$$

we have $\mu(Q_\rho) = \lim_{\nu\to\infty} \mu(\bigcup_{\sigma=\rho}^{\rho+\nu} P_\sigma)$, because $\bigcup_{\sigma=\rho}^{\rho+\nu} P_\sigma \uparrow_{\nu=1}^{\infty} Q_\rho$ for every $\rho = 1, 2, \ldots$. On the other hand, we conclude by the fact proved just above

$$\mu(\bigcup_{\sigma=\rho}^{\rho+\nu} P_\sigma) > \alpha - \sum_{\sigma=\rho}^{\rho+\nu} \frac{1}{2^\sigma}$$

Thus we obtain

$$\mu(Q_\rho) > \alpha - \sum_{\sigma=\rho}^{\infty} \frac{1}{2^\sigma} = \alpha - \frac{1}{2^{\rho-1}} .$$

Putting $P_0 = \bigwedge_{\rho=1}^{\infty} Q_\rho$, we have obviously $Q_\rho \downarrow_{\rho=1}^{\infty} P_0$, and hence

$$\mu(P_0) = \lim_{\rho\to\infty} \mu(Q_\rho) \geqq \alpha .$$

Consequently we obtain $\mu(P_0) = \alpha$. Furthermore we have for every $P \in \mathfrak{R}$

$$\mu(P_0 P) = \mu(P_0) - \mu(P_0(1-P)) = \alpha - \mu(P_0(1-P)) \geqq 0 ,$$
$$\mu((1-P_0)P) = \mu((1-P_0)P + P_0) - \mu(P_0)$$
$$= \mu((1-P_0)P + P_0) - \alpha \leqq 0 ,$$

because $(1-P_0)P\,P_0 = 0$, $(1-P_0)P + P_0 \in \mathfrak{R}$.

A positive totally additive function $\sigma(P)$ $(P \in \mathfrak{R})$ is called a majorant of an additive function $\mu(P)$ $(P \in \mathfrak{R})$, if

$$|\mu(P)| \leqq \sigma(P) \qquad \text{for every } P \in \mathfrak{R} .$$

If an additive function μ on $\mathfrak{R}$ has a majorant, then μ is obviously totally additive by definition.

Theorem 16.2. *Every totally additive function μ on $\mathfrak{R}$ has a majorant.*

Proof. If $\mu(P)$ $(P \in \mathfrak{R})$ is totally additive, then we see easily by definition that both $\mathfrak{R}\mu(P)$ and $\mathfrak{I}\mu(P)$ $(P \in \mathfrak{R})$ are totally additive. Therefore we can find by Theorem 16.1 $P_0, Q_0 \in \mathfrak{R}$ such that

$$\mathfrak{R}\mu(P_0 P) \geqq 0 , \quad \mathfrak{R}\mu((1-P_0)P) \leqq 0 ,$$
$$\mathfrak{I}\mu(Q_0 P) \geqq 0 , \quad \mathfrak{I}\mu((1-Q_0)P) \leqq 0$$

for every $P \in \mathfrak{R}$. For such $P_0, Q_0 \in \mathfrak{R}$, putting

$$\sigma(P) = \mathfrak{R}\mu(P_0 P) - \mathfrak{R}\mu((1-P_0)P) + \mathfrak{I}\mu(Q_0 P) - \mathfrak{I}\mu((1-Q_0)P),$$

we obtain obviously a positive totally additive function $\sigma(P)$ $(P \in \mathfrak{R})$. Furthermore we have for every $P \in \mathfrak{R}$

$$|\mu(P)| \leqq |\mathfrak{R}\mu(P)| + |\mathfrak{I}\mu(P)| = |\mathfrak{R}\mu(P_0 P) + \mathfrak{R}\mu((1-P_0)P)|$$
$$+ |\mathfrak{I}\mu(Q_0 P) + \mathfrak{I}\mu((1-Q_0)P)| \leqq \sigma(P).$$

Let $\mu(P)$ $(P \in \mathfrak{R})$ be a totally additive function with a majorant $\sigma(P)$ $(P \in \mathfrak{R})$. Let $\varphi(\xi)$ $(\xi \in \mathfrak{U}_{p_0})$ be a bounded continuous function on a neighbourhood $\mathfrak{U}_{p_0}$ of the proper space $\mathfrak{C}_{\mathfrak{R}}$. For any $\varepsilon > 0$, we can find a system $P_\lambda \in \mathfrak{R}$ $(\lambda \in \Lambda)$ such that

$$\sup_{\xi,\eta \in \mathfrak{U}_{p_\lambda}} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \qquad \text{for every } \lambda \in \Lambda ,$$

$$U_{P_0} = \sum_{\lambda \in \Lambda} U_{P_\lambda}.$$

Since U_{P_0} is compact by Theorem 15.4, we can find a finite number of elements $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots, \varkappa)$ such that

$$U_{P_0} = \sum_{\nu=1}^{\varkappa} U_{P_{\lambda_\nu}}.$$

For such P_{λ_ν} $(\nu = 1, 2, \ldots, \varkappa)$, putting

$$Q_1 = P_{\lambda_1}, \qquad Q_\nu = P_{\lambda_\nu}\Big(1 - \bigcup_{\rho=1}^{\nu-1} P_{\lambda_\rho}\Big) \quad (\nu = 2, 3, \ldots),$$

we obtain a <u>partition</u> of P_0 in $\mathfrak{R}$: $P_0 = \sum_{\nu=1}^{\varkappa} Q_\nu$, $Q_\nu \in \mathfrak{R}$ $(\nu = 1, 2, \ldots, \varkappa)$ such that

$$\sup_{\xi, \eta \in U_{Q_\nu}} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \qquad (\nu = 1, 2, \ldots, \varkappa).$$

Here $P_0 = \sum_{\nu=1}^{\varkappa} Q_\nu$, $Q_\nu \in \mathfrak{R}$ $(\nu = 1, 2, \ldots, \varkappa)$, implies $Q_\nu Q_\mu = 0$ for $\nu \neq \mu$. Because from $P_0 P_0 = P_0$ we conclude $\sum_{\nu \neq \rho} Q_\nu Q_\rho = 0$, and hence for every $x \in \mathfrak{H}$

$$\sum_{\nu \neq \rho} \| Q_\nu Q_\rho x \|^2 = \sum_{\nu \neq \rho} (Q_\nu Q_\rho x, x) = \Big(\sum_{\nu \neq \rho} Q_\nu Q_\rho x, x\Big) = 0.$$

Thus we obtain $Q_\nu Q_\rho = 0$ for $\nu \neq \rho$.

For every two partitions of P_0 in $\mathfrak{R}$:

$$P_0 = \sum_{\nu=1}^{\varkappa} P_\nu, \qquad \sup_{\xi, \eta \in U_{P_\nu}} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon,$$

$$P_0 = \sum_{\rho=1}^{\lambda} Q_\rho, \qquad \sup_{\xi, \eta \in U_{Q_\rho}} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon,$$

and for every $\xi_\nu \in U_{P_\nu}$, $\eta_\rho \in U_{Q_\rho}$ $(\nu = 1, 2, \ldots, \varkappa;\ \rho = 1, 2, \ldots, \lambda)$ we have

$$\sum_{\nu=1}^{\varkappa} \varphi(\xi_\nu)\mu(P_\nu) - \sum_{\rho=1}^{\lambda} \varphi(\eta_\rho)\mu(Q_\rho) = \sum_{\nu, \rho} \{\varphi(\xi_\nu) - \varphi(\eta_\rho)\}\mu(P_\nu Q_\rho),$$

because $P_\nu = \sum_{\rho=1}^{\lambda} P_\nu Q_\rho$, $Q_\rho = \sum_{\nu=1}^{\varkappa} P_\nu Q_\rho$. If $P_\nu Q_\rho \neq 0$, then we have $U_{P_\nu} U_{Q_\rho} \neq 0$ by §15(5), and hence for such ν, ρ

$$|\varphi(\xi_\nu) - \varphi(\eta_\rho)| \leqq 2\varepsilon.$$

Thus we obtain

$$\Big|\sum_{\nu=1}^{\varkappa} \varphi(\xi_\nu)\mu(P_\nu) - \sum_{\rho=1}^{\lambda} \varphi(\eta_\rho)\mu(Q_\rho)\Big| \leqq 2\varepsilon \sum_{\nu, \rho} |\mu(P_\nu Q_\rho)| \leqq 2\varepsilon \sum_{\nu, \rho} \sigma(P_\nu Q_\rho) = 2\varepsilon\sigma(P_0),$$

because $P_0 = \sum_{\nu, \rho} P_\nu Q_\rho$. Therefore there exists

$$\lim_{\varepsilon \to 0} \sum_{\nu=1}^{\varkappa} \varphi(\xi_\nu)\mu(P_\nu)$$

for $P_0 = \sum_{\nu=1}^{\varkappa} P_\nu$, $\xi_\nu \in U_{P_\nu}$, $\sup_{\xi, \eta \in U_{P_\nu}} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon$.

This limit is called the integral of φ by μ in U_{P_0} and denoted by

$$\int_{P_0} \varphi(\xi)\mu(d\xi).$$

With this definition we see easily

(1) $$\int_P \varphi(\xi)\mu(d\xi) + \int_Q \varphi(\xi)\mu(d\xi) = \int_{P+Q} \varphi(\xi)\mu(d\xi) \quad \text{for } PQ = 0,$$

(2) $$\int_P \{\alpha\varphi(\xi) + \beta\psi(\xi)\}\mu(d\xi) = \alpha\int_P \varphi(\xi)\mu(d\xi) + \beta\int_P \psi(\xi)\mu(d\xi),$$

(3) $$\int_P \mu(d\xi) = \mu(P),$$

(4) $$\int_{PQ} \varphi(\xi)\mu(d\xi) = \int_P \varphi(\xi)\mu(Qd\xi),$$

(5) $$\int_P \varphi(\xi)(\alpha\mu_1(d\xi) + \beta\mu_2(d\xi)) = \alpha\int_P \varphi(\xi)\mu_1(d\xi) + \beta\int_P \varphi(\xi)\mu_2(d\xi),$$

(6) $$\left|\int_P \varphi(\xi)\mu(d\xi)\right| \leqq \int_P \psi(\xi)\sigma(d\xi) \quad \text{if } |\varphi(\xi)| \leqq \psi(\xi) \text{ in } U_P.$$

Theorem 16.3. $\mathcal{R} \ni P_\nu \downarrow_{\nu=1}^{\infty} P_0$ or $P_\nu \uparrow_{\nu=1}^{\infty} P_0$ implies

$$\lim_{\nu\to\infty} \int_{P_\nu} \varphi(\xi)\mu(d\xi) = \int_{P_0} \varphi(\xi)\mu(d\xi).$$

Proof. If $\mathcal{R} \ni P_\nu \downarrow_{\nu=1}^{\infty} 0$, then, putting $\gamma = \sup_{\xi \in U_{P_1}} |\varphi(\xi)|$, we have by (6) and (3) for every $\nu = 1, 2, \ldots$

$$\left|\int_{P_\nu} \varphi(\xi)\mu(d\xi)\right| \leqq \int_{P_\nu} \gamma\,\sigma(d\xi) = \gamma\sigma(P_\nu),$$

and hence

$$\lim_{\nu\to\infty} \int_{P_\nu} \varphi(\xi)\mu(d\xi) = 0.$$

If $\mathcal{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P_0$, then, since $P_0 - P_\nu \downarrow_{\nu=1}^{\infty} 0$, we have by (1)

$$\lim_{\nu\to\infty} \left\{\int_{P_0} \varphi(\xi)\mu(d\xi) - \int_{P_\nu} \varphi(\xi)\mu(d\xi)\right\} = \lim_{\nu\to\infty} \int_{P_0 - P_\nu} \varphi(\xi)\mu(d\xi) = 0,$$

as proved just above. We also can prove likewise the other case: $\mathcal{R} \ni P_\nu \downarrow_{\nu=1}^{\infty} P_0$.

A function $\varphi(\xi)$ $(\xi \in U_{P_0})$ is said to be almost continuous,

if we can find $\mathcal{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P_0$ such that φ is bounded continuous in $\mathfrak{T}_{P_\nu}$ for every $\nu = 1, 2, \ldots$. Let φ be a almost continuous function on $\mathfrak{T}_{P_0}$. If for every $\mathcal{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P_0$ such that φ is bounded continuous in $\mathfrak{T}_{P_\nu}$ for every $\nu = 1, 2, \ldots$, we obtain the same limit

$$\lim_{\nu\to\infty} \int_{P_\nu} \varphi(\xi)\mu(d\xi),$$

then we shall say that φ is integrable by μ in $\mathfrak{T}_{P_0}$ and this same limit will be denoted by

$$\int_{P_0} \varphi(\xi)\mu(d\xi).$$

With this definition we see by Theorem 16.3 that every bounded continuous function on $\mathfrak{T}_{P_0}$ is integrable by μ in $\mathfrak{T}_{P_0}$ and we obtain the same integral.

Theorem 16.4. For a majorant $\sigma(P)$ of $\mu(P)$ $(P \in \mathcal{R})$, if we can find $\mathcal{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P_0$ such that a function φ is bounded continuous in $\mathfrak{T}_{P_\nu}$ for every $\nu = 1, 2, \ldots$, and

$$\lim_{\nu\to\infty} \int_{P_\nu} |\varphi(\xi)|\sigma(d\xi) = \alpha < +\infty,$$

then φ is integrable by μ in $\mathfrak{T}_{P_0}$ and

$$\left|\int_{P_0} \varphi(\xi)\mu(d\xi)\right| \leqq \alpha.$$

Proof. For every $\mathcal{R} \ni Q_\rho \uparrow_{\rho=1}^{\infty} P_0$ such that φ is bounded continuous in $\mathfrak{T}_{Q_\rho}$ for every $\rho = 1, 2, \ldots$, since $P_\nu Q_\rho \uparrow_{\nu=1}^{\infty} Q_\rho$, we have by Theorem 16.3 and the formula (1) for every $\rho = 1, 2, \ldots$

$$\begin{aligned}\int_{Q_\rho} |\varphi(\xi)|\sigma(d\xi) &= \lim_{\nu\to\infty} \int_{Q_\rho P_\nu} |\varphi(\xi)|\sigma(d\xi) \\ &\leqq \lim_{\nu\to\infty} \int_{P_\nu} |\varphi(\xi)|\sigma(d\xi) = \alpha.\end{aligned}$$

From $P_\nu Q_\rho \uparrow_{\rho=1}^{\infty} P_\nu$ we conclude likewise

$$\int_{P_\nu} |\varphi(\xi)|\sigma(d\xi) \leqq \lim_{\rho\to\infty} \int_{Q_\rho} |\varphi(\xi)|\sigma(d\xi).$$

Thus we have

$$\lim_{\rho\to\infty} \int_{Q_\rho} |\varphi(\xi)|\sigma(d\xi) = \alpha.$$

Furthermore, since $P_\nu Q_\nu \uparrow_{\nu=1}^{\infty} P_0$ by Theorem 13.8, we also obtain hence

$$\lim_{\nu\to\infty} \int_{P_\nu Q_\nu} |\varphi(\xi)|\sigma(d\xi) = \alpha.$$

For $\rho > \nu$ we have by the formulas (1) and (6)

$$\begin{aligned}
&\Big|\int_{Q_\rho}\varphi(\mathfrak{x})\mu(d\mathfrak{x}) - \int_{P_\nu}\varphi(\mathfrak{x})\mu(d\mathfrak{x})\Big| \\
&\quad\leqq \Big|\int_{Q_\rho - P_\nu Q_\rho}\varphi(\mathfrak{x})\mu(d\mathfrak{x})\Big| + \Big|\int_{P_\nu - P_\nu Q_\rho}\varphi(\mathfrak{x})\mu(d\mathfrak{x})\Big| \\
&\quad\leqq \int_{Q_\rho - P_\nu Q_\rho}|\varphi(\mathfrak{x})|\sigma(d\mathfrak{x}) + \int_{P_\nu - P_\nu Q_\rho}|\varphi(\mathfrak{x})|\sigma(d\mathfrak{x}) \\
&\quad\leqq 2\alpha - 2\int_{P_\nu Q_\nu}|\varphi(\mathfrak{x})|\sigma(d\mathfrak{x}).
\end{aligned}$$

Thus we obtain

$$\lim_{\nu,\rho\to\infty}\Big|\int_{Q_\rho}\varphi(\mathfrak{x})\mu(d\mathfrak{x}) - \int_{P_\nu}\varphi(\mathfrak{x})\mu(d\mathfrak{x})\Big| = 0.$$

From this relation we conclude easily

$$\lim_{\rho\to\infty}\int_{Q_\rho}\varphi(\mathfrak{x})\mu(d\mathfrak{x}) = \lim_{\nu\to\infty}\int_{P_\nu}\varphi(\mathfrak{x})\mu(d\mathfrak{x}).$$

Theorem 16.5. *If a function φ is integrable by μ in $\mathfrak{U}_{P_0}$, then φ also is integrable by μ in $\mathfrak{U}_P$ for every $\mathfrak{R} \ni P \leqq P_0$, and*

$$\int_{P_0}\varphi(\mathfrak{x})\mu(d\mathfrak{x}) = \int_{P}\varphi(\mathfrak{x})\mu(d\mathfrak{x}) + \int_{P_0 - P}\varphi(\mathfrak{x})\mu(d\mathfrak{x}).$$

Proof. For every two sequences $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P$ and $\mathfrak{R} \ni Q_\rho \uparrow_{\rho=1}^{\infty} P_0 - P$ such that φ is bounded continuous in $\mathfrak{U}_{P_\nu}$ and in $\mathfrak{U}_{Q_\rho}$ for every $\nu, \rho = 1, 2, \ldots$. Since $P_\nu + Q_{\rho_\nu} \uparrow_{\nu=1}^{\infty} P_0$, we have for every subsequence ρ_ν $(\nu = 1, 2, \ldots)$ by (1)

$$\int_{P_0}\varphi(\mathfrak{x})\mu(d\mathfrak{x}) = \lim_{\nu\to\infty}\Big\{\int_{P_\nu}\varphi(\mathfrak{x})\mu(d\mathfrak{x}) + \int_{Q_{\rho_\nu}}\varphi(\mathfrak{x})\mu(d\mathfrak{x})\Big\}.$$

Since the subsequence ρ_ν $(\nu = 1, 2, \ldots)$ may be arbitrary, we obtain hence

$$\int_{P_0}\varphi(\mathfrak{x})\mu(d\mathfrak{x}) = \lim_{\nu\to\infty}\int_{P_\nu}\varphi(\mathfrak{x})\mu(d\mathfrak{x}) + \lim_{\rho\to\infty}\int_{Q_\rho}\varphi(\mathfrak{x})\mu(d\mathfrak{x}).$$

From this relation we conclude easily our assertion.

§17 Integration by Elements

Let $\varphi(\mathfrak{x})$ $(\mathfrak{x} \in \mathfrak{U}_{P_0})$ be a bounded continuous function on a neighbourhood $\mathfrak{U}_{P_0}$ of the proper space $\mathfrak{E}_{\mathfrak{R}}$. For two partitions of P_0 in $\mathfrak{R}$:

$$P_0 = \sum_{\nu=1}^{\varkappa} P_\nu, \qquad \sup_{\mathfrak{x},\mathfrak{y}\in\mathfrak{U}_{P_\nu}}|\varphi(\mathfrak{x}) - \varphi(\mathfrak{y})| \leqq \varepsilon \quad (\nu = 1, 2, \ldots, \varkappa),$$

$$P_0 = \sum_{\rho=1}^{\lambda} Q_\rho, \qquad \sup_{\xi,\eta \in U_{Q_\rho}} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \quad (\rho = 1, 2, \ldots, \lambda),$$

and for $\xi_\nu \in U_{P_\nu}$, $\eta_\rho \in U_{Q_\rho}$, we have for every $a \in R$

$$\begin{aligned} \| \sum_{\nu=1}^{\kappa} \varphi(\xi_\nu) P_\nu a - \sum_{\rho=1}^{\lambda} \varphi(\eta_\rho) Q_\rho a \| &= \sum_{\nu,\rho} |\varphi(\xi_\nu) - \varphi(\eta_\rho)|^2 \| P_\nu Q_\rho a \|^2 \\ &\leqq 4\varepsilon^2 \sum_{\nu,\rho} \| P_\nu Q_\rho a \|^2 = 4\varepsilon^2 \| P_0 a \|^2. \end{aligned}$$

Therefore there exists

$$\lim_{\varepsilon \to 0} \sum_{\nu=1}^{\kappa} \varphi(\xi_\nu) P_\nu a$$

for $P_0 = \sum_{\nu=1}^{\kappa} P_\nu$, $\sup_{\xi,\eta \in U_{P_\nu}} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon$, $\xi_\nu \in U_{P_\nu}$ $(\nu = 1, 2, \ldots, \kappa)$.

This limit is called the integral of φ by an element a in U_{P_0} and denoted by

$$\int_P \varphi(\xi)\, d_\xi a.$$

With this definition we have obviously

Theorem 17.1. *For a bounded continuous function φ on U_P, we have*

$$\| \sum_{\nu=1}^{\kappa} \varphi(\xi_\nu) P_\nu a - \int_P \varphi(\xi)\, d_\xi a \| \leqq 2\varepsilon \| Pa \|$$

for $P = \sum_{\nu=1}^{\kappa} P_\nu$, $\sup_{\xi,\eta \in U_{P_\nu}} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon$, $\xi_\nu \in U_{P_\nu}$ $(\nu = 1, 2, \ldots, \kappa)$.

Furthermore we see easily

(1) $$\int_P \varphi(\xi)\, d_\xi a + \int_Q \varphi(\xi)\, d_\xi a = \int_{P+Q} \varphi(\xi)\, d_\xi a \quad \text{for } PQ = 0,$$

(2) $$\int_P \{\alpha\varphi(\xi) + \beta\psi(\xi)\}\, d_\xi a = \alpha \int_P \varphi(\xi)\, d_\xi a + \beta \int_P \psi(\xi)\, d_\xi a,$$

(3) $$\int_P d_\xi a = Pa,$$

(4) $$Q \int_P \varphi(\xi)\, d_\xi a = \int_{PQ} \varphi(\xi)\, d_\xi a = \int_P \varphi(\xi)\, d_\xi Qa,$$

(5) $$\int_P \varphi(\xi)\, d_\xi (\alpha a + \beta b) = \alpha \int_P \varphi(\xi)\, d_\xi a + \beta \int_P \varphi(\xi)\, d_\xi b.$$

For every $a, b \in R$, (Pa, b) $(P \in \mathfrak{R})$ is obviously a tota-

lly additive function on the ring $\mathfrak{R}$, and we have obviously by Theorem 17.1

$$(6)\qquad \left(\int_{\mathfrak{p}} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} a, \xi\right) = \int_{\mathfrak{p}} \varphi(\mathfrak{x})(d_{\mathfrak{x}} a, \xi).$$

$$(7)\qquad \left(\int_{\mathfrak{p}} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a,\ \int_{\mathfrak{p}} \psi(\mathfrak{x}) d_{\mathfrak{x}} \xi\right) = \int_{\mathfrak{p}} \varphi(\mathfrak{x})\overline{\psi(\mathfrak{x})}(d_{\mathfrak{x}} a, \xi),$$

because we have by definition

$$\left(\int_{\mathfrak{p}_0} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a,\ \int_{\mathfrak{p}_0} \psi(\mathfrak{x}) d_{\mathfrak{x}} \xi\right) = \lim_{\varepsilon \to 0} \left(\sum_{\nu=1}^{\kappa} \varphi(\mathfrak{x}_\nu) P_\nu a, \sum_{\nu=1}^{\kappa} \psi(\mathfrak{x}_\nu) P_\nu \xi\right)$$
$$= \lim_{\varepsilon \to 0} \sum_{\nu=1}^{\kappa} \varphi(\mathfrak{x}_\nu)\overline{\psi(\mathfrak{x}_\nu)}(P_\nu a, \xi) = \int_{\mathfrak{p}_0} \varphi(\mathfrak{x})\overline{\psi(\mathfrak{x})}(d_{\mathfrak{x}} a, \xi).$$

Especially we have

$$(8)\qquad \left(\int_{\mathfrak{p}} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a,\ \int_{\mathfrak{p}} \psi(\mathfrak{x}) d_{\mathfrak{x}} a\right) = \int_{\mathfrak{p}} \varphi(\mathfrak{x})\overline{\psi(\mathfrak{x})} \|d_{\mathfrak{x}} a\|^2.$$

$$(9)\qquad \left\|\int_{\mathfrak{p}} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a\right\|^2 = \int_{\mathfrak{p}} |\varphi(\mathfrak{x})|^2 \|d_{\mathfrak{x}} a\|^2.$$

Here $\|Pa\|^2$ $(P \in \mathfrak{R})$ is obviously a positive totally additive function on $\mathfrak{R}$.

For a bounded linear operator T on $\mathfrak{R}$, we obtain at once by definition

$$(10)\qquad T\int_{\mathfrak{p}} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a = \int_{\mathfrak{p}} \varphi(\mathfrak{x}) d_{\mathfrak{x}} Ta$$

if T is commutative to $\mathfrak{R}$.

Theorem 17.2. If $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P$ or $P_\nu \downarrow_{\nu=1}^{\infty} P$, then we have

$$\lim_{\nu \to \infty} \int_{\mathfrak{p}_\nu} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a = \int_{\mathfrak{p}} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a.$$

Proof. If $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P$, then we have by (4)

$$\int_{\mathfrak{p}_\nu} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a = P_\nu \int_{\mathfrak{p}} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a,$$

and hence

$$\lim_{\nu \to \infty} \int_{\mathfrak{p}_\nu} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a = P \int_{\mathfrak{p}} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a = \int_{\mathfrak{p}} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a.$$

We also can prove likewise the other case: $\mathfrak{R} \ni P_\nu \downarrow_{\nu=1}^{\infty} P$.

Let $\varphi(\mathfrak{x})$ be an almost continuous function on $\mathfrak{O}_{\mathfrak{p}_0}$. If we can find a sequence $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P_0$ such that φ is bounded continuous in $\mathfrak{O}_{\mathfrak{p}_\nu}$ for every $\nu = 1, 2, \ldots$, and there exists the limit

$$\lim_{\nu \to \infty} \int_{\mathfrak{p}} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a,$$

then for every sequence $\mathfrak{R} \ni Q_\nu \uparrow_{\nu=1}^{\infty} P_0$ such that φ is bounded continuous in $\mathcal{U}_{Q_\nu}$ for every $\nu = 1, 2, \ldots$, we have always

$$\lim_{\nu\to\infty} \int_{P_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a = \lim_{\nu\to\infty} \int_{Q_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a .$$

In fact, we have by (4) for every ν, $\mu = 1, 2, \ldots$

$$P_\nu \int_{Q_\mu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a = Q_\mu \int_{P_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a,$$

and hence

$$\int_{Q_\mu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a = P_0 \int_{Q_\mu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a = Q_\mu \lim_{\nu\to\infty} \int_{P_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a$$

Thus, making μ tend to $+\infty$, we obtain

$$\lim_{\mu\to\infty} \int_{Q_\mu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a = P_0 \lim_{\nu\to\infty} \int_{P_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a = \lim_{\nu\to\infty} \int_{P_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a.$$

Therefore in this case we shall say that φ is <u>integrable</u> by a in $\mathcal{U}_{P_0}$ and this coincided limit is defined as the integral

$$\int_{P_0} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a .$$

<u>Theorem</u> 17.3. <u>An almost continuous function</u> φ <u>on</u> $\mathcal{U}_{P_0}$ <u>is integrable by an element</u> a <u>in</u> $\mathcal{U}_{P_0}$, <u>if and only if</u> $|\varphi(\mathfrak{x})|^2$ <u>is integrable by</u> $\|Pa\|^2$ $(P \in \mathfrak{R})$ <u>in</u> $\mathcal{U}_{P_0}$, <u>and then we have</u>

$$\Big\| \int_{P_0} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a \Big\|^2 = \int_{P_0} |\varphi(\mathfrak{x})|^2 \, \| d_{\mathfrak{x}}\, a \|^2 .$$

<u>Proof</u>. If $|\varphi(\mathfrak{x})|^2$ is integrable by $\|Pa\|^2$ $(P \in \mathfrak{R})$ in $\mathcal{U}_{P_0}$, then we can find by definition a sequence $\mathfrak{R} \ni Q_\nu \uparrow_{\nu=1}^{\infty} P_0$ such that φ is bounded continuous in $\mathcal{U}_{Q_\nu}$ for every $\nu = 1, 2, \ldots$ and

$$\sup_{\nu \geq 1} \int_{Q_\nu} |\varphi(\mathfrak{x})|^2 \, \| d_{\mathfrak{x}}\, a \|^2 < +\infty .$$

For such Q_ν $(\nu = 1, 2, \ldots)$, putting $Q_0 = 0$, we have by (9) and (4)

$$\begin{aligned} \int_{Q_\nu} |\varphi(\mathfrak{x})|^2 \, \| d_{\mathfrak{x}}\, a \|^2 &= \Big\| \int_{Q_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a \Big\|^2 \\ &= \Big\| \sum_{\mu=1}^{\nu} (Q_\mu - Q_{\mu-1}) \int_{Q_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a \Big\|^2 \\ &= \sum_{\mu=1}^{\nu} \Big\| (Q_\mu - Q_{\mu-1}) \int_{Q_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a \Big\|^2 = \sum_{\mu=1}^{\nu} \Big\| \int_{Q_\mu - Q_{\mu-1}} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\, a \Big\|^2, \end{aligned}$$

because $(Q_\mu - Q_{\mu-1})(Q_\nu - Q_{\nu-1}) = 0$ for $\mu \neq \nu$. Recalling Theorem 7.11, we can find hence by assumption $\ell \in R$ for which

$$\mathfrak{b} = \sum_{\mu=1}^{\infty} \int_{Q_\mu - Q_{\mu-1}} \varphi(\mathfrak{x}) d\mathfrak{x}\, a = \lim_{\nu\to\infty} \int_{Q_\nu} \varphi(\mathfrak{x}) d\mathfrak{x}\, a.$$

Therefore φ is integrable by a in $\mathfrak{o}_{P_0}$.

Conversely if φ is integrable by a in $\mathfrak{o}_{P_0}$, then we have by (9)

$$\lim_{\nu\to\infty} \int_{P_\nu} |\varphi(\mathfrak{x})|^2 \| d\mathfrak{x}\, a\|^2 = \lim_{\nu\to\infty} \| \int_{P_\nu} \varphi(\mathfrak{x}) d\mathfrak{x}\, a\|^2$$
$$= \| \int_{P_0} \varphi(\mathfrak{x}) d\mathfrak{x}\, a\|^2$$

for every sequence $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P_0$ subject to the condition that φ is bounded continuous in $\mathfrak{o}_{P_\nu}$ for every $\nu = 1, 2, \ldots$. Thus $|\varphi(\mathfrak{x})|^2$ is integrable by $\|Pa\|^2$ $(P \in \mathfrak{R})$ in $\mathfrak{o}_{P_0}$, and we have by definition

$$\int_{P_0} |\varphi(\mathfrak{x})|^2 \| d\mathfrak{x}\, a\|^2 = \| \int_{P_0} \varphi(\mathfrak{x}) d\mathfrak{x}\, a\|^2.$$

Theorem 17.4. If φ is integrable by a in $\mathfrak{o}_P$, then for every $Q \in \mathfrak{R}$, φ is integrable by a in $\mathfrak{o}_{PQ}$, by Qa in $\mathfrak{o}_P$, and

$$Q \int_P \varphi(\mathfrak{x}) d\mathfrak{x}\, a = \int_{PQ} \varphi(\mathfrak{x}) d\mathfrak{x}\, a = \int_P \varphi(\mathfrak{x}) d\mathfrak{x}\, Qa.$$

Proof. If φ is integrable by a in $\mathfrak{o}_{P_0}$, then $|\varphi(\mathfrak{x})|^2$ is by Theorem 17.3 integrable by $\|Pa\|^2$ $(P \in \mathfrak{R})$ in $\mathfrak{o}_{P_0}$. Recalling Theorem 16.4, we see then easily that $|\varphi(\mathfrak{x})|^2$ is integrable by $\|Pa\|^2$ $(P \in \mathfrak{R})$ in $\mathfrak{o}_{P_0 Q_0}$, and by $\|Q_0 P a\|^2$ $(P \in \mathfrak{R})$ in $\mathfrak{o}_{P_0}$ for every $Q_0 \in \mathfrak{R}$. Therefore φ is by Theorem 17.3 integrable by a in $\mathfrak{o}_{P_0 Q_0}$ and by $Q_0 a$ in $\mathfrak{o}_{P_0}$. Furthermore we can finish our assertion by the formula (4).

Theorem 17.5. If φ is integrable by a in $\mathfrak{o}_P$ as well as in $\mathfrak{o}_Q$, and $PQ = 0$, then φ also is integrable by a in $\mathfrak{o}_{P+Q}$, and

$$\int_P \varphi(\mathfrak{x}) d\mathfrak{x}\, a + \int_Q \varphi(\mathfrak{x}) d\mathfrak{x}\, a = \int_{P+Q} \varphi(\mathfrak{x}) d\mathfrak{x}\, a.$$

Proof. For two sequences $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P$ and $\mathfrak{R} \ni Q_\nu \uparrow_{\nu=1}^{\infty} Q$ such that φ is bounded continuous in $\mathfrak{o}_{P_\nu}$ as well as in $\mathfrak{o}_{Q_\nu}$ for every $\nu = 1, 2, \ldots$, recalling Theorem 17.3, we have by definition

$$\lim_{\nu\to\infty} \int_{P_\nu} |\varphi(\mathfrak{x})|^2 \| d\mathfrak{x}\, a\|^2 = \int_P |\varphi(\mathfrak{x})|^2 \| d\mathfrak{x}\, a\|^2 < +\infty,$$

$$\lim_{\nu\to\infty} \int_{Q_\nu} |\varphi(\mathfrak{x})|^2 \| d\mathfrak{x}\, a\|^2 = \int_Q |\varphi(\mathfrak{x})|^2 \| d\mathfrak{x}\, a\|^2 < +\infty,$$

and hence we obtain by the formula §16(1)

$$\lim_{\nu\to\infty} \int_{P_\nu+Q_\nu} |\varphi(\mathfrak{x})|^2 \| d_{\mathfrak{x}} a \|^2 < +\infty .$$

Since $P_\nu + Q_\nu \uparrow_{\nu=1}^{\infty} P + Q$, we see by Theorems 17.3 and 16.4 that φ is integrable by a in $\mathfrak{o}_{P+Q}$. Furthermore we obtain by Theorem 17.4

$$\begin{aligned}\int_P \varphi(\mathfrak{x}) d_{\mathfrak{x}} a + \int_Q \varphi(\mathfrak{x}) d_{\mathfrak{x}} a &= P \int_{P+Q} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a + Q \int_{P+Q} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a \\ &= \int_{P+Q} \varphi(\mathfrak{x}) d_{\mathfrak{x}} a .\end{aligned}$$

Theorem 17.6. *If both φ and ψ are integrable by a in $\mathfrak{o}_P$, then $\alpha\varphi(\mathfrak{x}) + \beta\psi(\mathfrak{x})$ is integrable by a in $\mathfrak{o}_P$ for every complex numbers α, β, and*

$$\int_P \{\alpha\varphi(\mathfrak{x}) + \beta\psi(\mathfrak{x})\} d_{\mathfrak{x}} a = \alpha \int_P \varphi(\mathfrak{x}) d_{\mathfrak{x}} a + \beta \int_P \psi(\mathfrak{x}) d_{\mathfrak{x}} a .$$

Proof. Since $|\alpha\varphi(\mathfrak{x}) + \beta\psi(\mathfrak{x})|^2 \leqq 2\{|\alpha|^2|\varphi(\mathfrak{x})|^2 + |\beta|^2|\psi(\mathfrak{x})|^2\}$, we conclude easily our assertion by Theorems 17.3, 16.4 and the formula (2).

By virtue of Theorems 17.3 and 16.4, we can conclude further from the formula (5)

Theorem 17.7. *If φ is integrable by a as well as by b in $\mathfrak{o}_P$, then φ also is integrable by $\alpha a + \beta b$ in $\mathfrak{o}_P$ for every complex numbers α, β and*

$$\int_P \varphi(\mathfrak{x}) d_{\mathfrak{x}} (\alpha a + \beta b) = \alpha \int_P \varphi(\mathfrak{x}) d_{\mathfrak{x}} a + \beta \int_P \varphi(\mathfrak{x}) d_{\mathfrak{x}} b .$$

Theorem 17.8. *If φ is integrable by every a_ν $(\nu = 1, 2, \ldots)$ in $\mathfrak{o}_P$ and*

$$\lim_{\nu\to\infty} a_\nu = a, \quad \lim_{\nu\to\infty} \int_P \varphi(\mathfrak{x}) d_{\mathfrak{x}} a_\nu = b,$$

then φ is integrable by a in $\mathfrak{o}_P$ and $\int_P \varphi(\mathfrak{x}) d_{\mathfrak{x}} a = b$.

Proof. For every sequence $\mathfrak{N} \ni P_\mu \uparrow_{\mu=1}^{\infty} P$ such that

bounded continuous in $\mathcal{O}_{P_\mu}$ for every $\mu = 1, 2, \ldots$, putting

$$\gamma_\mu = \sup_{\xi \in \mathcal{O}_{P_\mu}} |\varphi(\xi)| \qquad (\mu = 1, 2, \ldots),$$

we obtain by (5) and (9)

$$\left\| \int_{P_\mu} \varphi(\xi)\, d_\xi a - \int_{P_\mu} \varphi(\xi)\, d_\xi a_\nu \right\|^2 = \left\| \int_{P_\mu} \varphi(\xi)\, d_\xi (a - a_\nu) \right\|^2$$

$$= \int_{P_\mu} |\varphi(\xi)|^2 \| d_\xi (a - a_\nu) \|^2 \leq \gamma_\mu \| P_\mu (a - a_\nu) \|^2 \leq \gamma_\mu \| a - a_\nu \|^2,$$

and hence by Theorem 17.4

$$\int_{P_\mu} \varphi(\xi)\, d_\xi a = \lim_{\nu \to \infty} \int_{P_\mu} \varphi(\xi)\, d_\xi a_\nu = \lim_{\nu \to \infty} P_\mu \int_P \varphi(\xi)\, d_\xi a_\nu = P_\mu b.$$

Thus we have $\lim_{\mu \to \infty} \int_{P_\mu} \varphi(\xi)\, d_\xi a = P b$. Therefore φ is by definition integrable by a in $\mathcal{O}_P$, and we have by Theorem 17.4

$$\int_P \varphi(\xi)\, d_\xi a = P b = \lim_{\nu \to \infty} P \int_P \varphi(\xi)\, d_\xi a_\nu = \lim_{\nu \to \infty} \int_P \varphi(\xi)\, d_\xi a_\nu = b.$$

Theorem 17.9. If φ is integrable by a in $\mathcal{O}_{P_0}$ and ψ is integrable by b in $\mathcal{O}_{P_0}$, then $\varphi(\xi)\overline{\psi(\xi)}$ is integrable by (Pa, b) $(P \in \mathfrak{R})$ and

$$\left(\int_{P_0} \varphi(\xi)\, d_\xi a, \int_{P_0} \psi(\xi)\, d_\xi b \right) = \int_{P_0} \varphi(\xi)\overline{\psi(\xi)}\, (d_\xi a, b).$$

Proof. We can find by Theorem 13.8 a sequence $\mathfrak{R} \ni Q_\mu \uparrow_{\mu=1}^{\infty} P_0$ such that both φ and ψ are bounded continuous in $\mathcal{O}_{Q_\mu}$ for every $\mu = 1, 2, \ldots$. For every $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P_0$ such that $\varphi(\xi)\overline{\psi(\xi)}$ is bounded continuous in $\mathcal{O}_{P_\nu}$ for every $\nu = 1, 2, \ldots$, we have by (7)

$$\left(\int_{P_\nu Q_\mu} \varphi(\xi)\, d_\xi a, \int_{P_\nu Q_\mu} \psi(\xi)\, d_\xi b \right) = \int_{P_\nu Q_\mu} \varphi(\xi)\overline{\psi(\xi)}\, (d_\xi a, b).$$

Making μ tend to $+\infty$, we obtain by Theorems 16.3 and 17.4

$$\int_{P_\nu} \varphi(\xi)\overline{\psi(\xi)}\, (d_\xi a, b) = \left(\int_{P_\nu} \varphi(\xi)\, d_\xi a, \int_{P_\nu} \psi(\xi)\, d_\xi b \right)$$

$$= \left(P_\nu \int_{P_0} \varphi(\xi)\, d_\xi a, \int_{P_0} \psi(\xi)\, d_\xi b \right),$$

and hence

$$\lim_{\nu \to \infty} \int_{P_\nu} \varphi(\xi)\overline{\psi(\xi)}\, (d_\xi a, b) = \left(\int_{P_0} \varphi(\xi)\, d_\xi a, \int_{P_0} \psi(\xi)\, d_\xi b \right).$$

Therefore $\varphi(\xi)\overline{\psi(\xi)}$ is integrable by (Pa, b) $(P \in \mathfrak{R})$ in $\mathcal{O}_{P_0}$, and

$$\int_{P_0} \varphi(\xi)\overline{\psi(\xi)}\,(d_\xi a, b) = \left(\int_{P_0} \varphi(\xi) d_\xi a,\ \int_{P_0} \psi(\xi) d_\xi b\right).$$

Putting $\psi(\xi) = 1$ in this Theorem 17.9 we obtain

Theorem 17.10. If φ is integrable by a in $\mathfrak{U}_{P_0}$, then $\varphi(\xi)$ is integrable by (Pa, b) $(P \in \mathfrak{R})$ for every $b \in R$, and

$$\left(\int_{P_0} \varphi(\xi) d_\xi a, b\right) = \int_{P_0} \varphi(\xi)\,(d_\xi a, b).$$

Recalling the formula (10), we have obviously by definition

Theorem 17.11. If φ is integrable by a in $\mathfrak{U}_P$, then for any bounded linear operator T on R being commutative to $\mathfrak{R}$, φ also is integrable by Ta in $\mathfrak{U}_P$, and

$$T\int_P \varphi(\xi) d_\xi a = \int_P \varphi(\xi) d_\xi Ta.$$

§18 Relative Spectra

Let $\mu(P)$ $(P \in \mathfrak{R})$ be a function on a closed ring of projection operators $\mathfrak{R}$. For a maximal ideal $\mathfrak{x}$ in $\mathfrak{R}$, if there exists a complex number ξ such that

$$\inf_{P \in \mathfrak{x}} \left\{ \sup_{P \geq Q \in \mathfrak{x}} |\mu(Q) - \xi| \right\} = 0,$$

then we shall write

$$\lim_{P \to \mathfrak{x}} \mu(P) = \xi.$$

If $\sup_{P \in \mathfrak{x}} \left\{ \inf_{P \geq Q \in \mathfrak{x}} |\mu(Q)| \right\} = +\infty$, then we shall write

$$\lim_{P \to \mathfrak{x}} \mu(P) = \infty.$$

With this definition, we have obviously

Theorem 18.1. For two functions μ_1 and μ_2 on $\mathfrak{R}$

$$\lim_{P \to \mathfrak{x}} \mu_1(P) = \xi_1, \qquad \lim_{P \to \mathfrak{x}} \mu_2(P) = \xi_2$$

implies

$$\lim_{P \to \mathfrak{x}} (\alpha\mu_1(P) + \beta\mu_2(P)) = \alpha\xi_1 + \beta\xi_2,$$

$$\lim_{P \to \mathfrak{x}} \mu_1(P)\mu_2(P) = \xi_1\xi_2,$$

$$\lim_{P \to \mathfrak{x}} \frac{\mu_1(P)}{\mu_2(P)} = \frac{\xi_1}{\xi_2}$$

if the right side has any sense.

A positive function $\mu(P)$ $(P \in \mathcal{R})$ is said to be proper in $P_0 \in \mathcal{R}$, if $\mu(P) = 0$ implies $P P_0 = 0$. With this definition we have

Theorem 18.2. For two totally additive functions μ and σ on $\mathcal{R}$, if σ is positive and proper in $P_0 \in \mathcal{R}$, then, putting

$$\varphi(\mathfrak{P}) = \lim_{P \to \mathfrak{P}} \frac{\mu(P)}{\sigma(P)} \qquad \text{for every } \mathfrak{P} \in \mathfrak{U}_{P_0},$$

we obtain a continuous function φ on $\mathfrak{U}_{P_0}$, and

$$\mu(P) = \int_P \varphi(\mathfrak{P}) \sigma(d\mathfrak{P}) \qquad \text{for } \mathcal{R} \ni P \leqq P_0.$$

Proof. We consider firstly the case where μ is a real valued function on $\mathcal{R}$. For every real number ξ

$$\mu(P) - \xi \sigma(P) \qquad (P \in \mathcal{R})$$

is obviously a totally additive function on $\mathcal{R}$. By virtue of Theorem 16.1, corresponding to every real number ξ, we obtain hence a projection operator $Q_\xi \in \mathcal{R}$ such that

$$\mu(P) - \xi \sigma(P) \begin{cases} \geqq 0 & \text{for } \mathcal{R} \ni P \leqq Q_\xi \\ \leqq 0 & \text{for } \mathcal{R} \ni P \leqq 1 - Q_\xi. \end{cases}$$

For such Q_ξ $(-\infty < \xi < +\infty)$, we have

$$P_0 Q_\xi \leqq P_0 Q_\eta \qquad \text{for } \xi > \eta.$$

Because, putting $P = Q_\xi(1 - Q_\eta)$, we have

$$\mu(P) - \xi \sigma(P) \geqq 0, \qquad \mu(P) - \eta \sigma(P) \leqq 0,$$

and hence $(\xi - \eta)\sigma(P) \leqq 0$. Since $\xi > \eta$, we obtain therefore $\sigma(P) = 0$. Since σ is proper in P_0 by assumption, we conclude hence $P P_0 = 0$, and consequently $P_0 Q_\xi = P_0 Q_\xi Q_\eta \leqq P_0 Q_\eta$.

Putting

$$\varphi(\mathfrak{P}) = \begin{cases} \sup_{Q_\xi \in \mathfrak{P}} \xi & \text{for } \mathfrak{P} \in \sum_{-\infty < \xi < +\infty} \mathfrak{U}_{Q_\xi} \\ -\infty & \text{for } \mathfrak{P} \bar{\in} \sum_{-\infty < \xi < +\infty} \mathfrak{U}_{Q_\xi}, \end{cases}$$

we obtain a function on the proper space $\mathfrak{L}_{\mathcal{R}}$. For a point $\mathfrak{P}_0 \in \mathfrak{U}_{P_0}$, if $\varphi(\mathfrak{P}_0) > \eta$, then there exists $\xi > \eta$ such that $Q_\xi \in \mathfrak{P}_0$, and it is obvious that

$$\varphi(\mathfrak{P}) \geqq \xi \qquad \text{for every } \mathfrak{P} \in \mathfrak{U}_{Q_\xi P_0}.$$

If $\varphi(\mathfrak{P}_0) < \xi$, then we have $Q_\xi \bar{\in} \mathfrak{P}_0$, and hence $\mathfrak{P}_0 \in \mathfrak{U}_{(1-Q_\xi)P_0}$.

Furthermore we have

$$\varphi(\mathfrak{x}) \leqq \xi \qquad \text{for every } \mathfrak{x} \in \mathfrak{U}_{(1-Q_\xi)P_0},$$

because for $\xi' > \xi$ we have $P_0 Q_{\xi'} \leqq P_0 Q_\xi$, and hence

$$\mathfrak{U}_{(1-Q_\xi)P_0}\,\mathfrak{U}_{Q_{\xi'}} = \mathfrak{U}_{(1-Q_\xi)P_0 Q_{\xi'}} = 0.$$

Therefore φ is continuous in $\mathfrak{U}_{P_0}$. Since we have

$$\frac{\mu(P)}{\sigma(P)} \begin{cases} \geqq \xi & \text{for } 0 \neq P \leqq Q_\xi P_0 \\ \leqq \zeta & \text{for } 0 \neq P \leqq (1-Q_\zeta)P_0, \end{cases}$$

we conclude further

$$\varphi(\mathfrak{x}) = \lim_{P \to \mathfrak{x}} \frac{\mu(P)}{\sigma(P)} \qquad \text{for every } \mathfrak{x} \in \mathfrak{U}_{P_0}.$$

Putting $P = \bigcap_{\nu=1}^{\infty} Q_\nu P_0$, we have obviously

$$\mu(P) - \nu\sigma(P) \geqq 0 \qquad \text{for every } \nu = 1, 2, \ldots,$$

and hence $\sigma(P) = 0$. Since σ is proper in P_0 by assumption, we obtain thus $P = 0$, and consequently

$$(1-Q_\nu)P_0 \uparrow_{\nu=1}^{\infty} P_0.$$

Putting $P = \bigcap_{\nu=1}^{\infty} (1-Q_{-\nu})P_0$, we have obviously

$$\mu(P) + \nu\sigma(P) \leqq 0 \qquad \text{for every } \nu = 1, 2, \ldots,$$

and hence $P = 0$. This relation yields likewise $Q_{-\nu} P_0 \uparrow_{\nu=1}^{\infty} P_0$. Consequently we obtain by Theorem 13.8

$$Q_{-\nu}(1-Q_\nu)P_0 \uparrow_{\nu=1}^{\infty} P_0.$$

On the other hand, $\mathfrak{R} \ni P \leqq Q_{-\nu}(1-Q_\nu)P_0$ implies

$$-\nu \leqq \frac{\mu(P)}{\sigma(P)} \leqq \nu$$

and hence $|\varphi(\mathfrak{x})| \leqq \nu$ for $\mathfrak{x} \in \mathfrak{U}_{Q_{-\nu}(1-Q_\nu)P_0}$. Therefore φ is almost continuous in $\mathfrak{U}_{P_0}$.

If φ is bounded in $\mathfrak{U}_P$ for some $\mathfrak{R} \ni P \leqq P_0$, and

$$\alpha < \varphi(\mathfrak{x}) < \beta \qquad \text{for every } \mathfrak{x} \in \mathfrak{U}_P,$$

then we have $P \leqq (Q_\alpha - Q_\beta)P_0$, because

$$\varphi(\mathfrak{x}) \begin{cases} \geqq \beta & \text{for } \mathfrak{x} \in \mathfrak{U}_{Q_\beta P_0} \\ \leqq \alpha & \text{for } \mathfrak{x} \in \mathfrak{U}_{(1-Q_\alpha)P_0}, \end{cases}$$

as proved just above. For

$$\alpha = \lambda_0 < \lambda_1 < \cdots < \lambda_\kappa = \beta, \quad \lambda_\nu - \lambda_{\nu-1} < \varepsilon \qquad (\nu = 1, 2, \ldots, \kappa)$$

we obtain hence a partition of P in $\mathfrak{R}$:

$$P = \sum_{\nu=1}^{\kappa} (Q_{\lambda_{\nu-1}} - Q_{\lambda_\nu})P,$$

such that

$$\lambda_\nu \sigma((Q_{\lambda_{\nu-1}} - Q_{\lambda_\nu})P) \geqq \mu((Q_{\lambda_{\nu-1}} - Q_{\lambda_\nu})P) \geqq \lambda_{\nu-1}\sigma((Q_{\lambda_{\nu-1}} - Q_{\lambda_\nu})P),$$

$$\lambda_\nu \geqq \varphi(\xi) \geqq \lambda_{\nu-1} \quad \text{for } \xi \in U_{(Q_{\lambda_{\nu-1}} - Q_{\lambda_\nu})P}.$$

Therefore, for $\xi_\nu \in U_{(Q_{\lambda_{\nu-1}} - Q_{\lambda_\nu})P}$ we have

$$\left|\sum_{\nu=1}^{N} \mu((Q_{\lambda_{\nu-1}} - Q_{\lambda_\nu})P) - \sum_{\nu=1}^{N} \varphi(\xi_\nu)\sigma((Q_{\lambda_{\nu-1}} - Q_{\lambda_\nu})P)\right| \leqq \varepsilon \sum_{\nu=1}^{N} \sigma((Q_{\lambda_{\nu-1}} - Q_{\lambda_\nu})P),$$

and hence

$$\left|\mu(P) - \sum_{\nu=1}^{N} \varphi(\xi_\nu)\sigma((Q_{\lambda_{\nu-1}} - Q_{\lambda_\nu})P)\right| \leqq \varepsilon\, \sigma(P).$$

This relation yields by definition

$$\mu(P) = \int_P \varphi(\xi)\sigma(d\xi).$$

For every $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P \leqq P_0$, such that φ is bounded in U_{P_ν} for every $\nu = 1, 2, \ldots$, we have

$$\mu(P_\nu) = \int_{P_\nu} \varphi(\xi)\sigma(d\xi) \qquad (\nu = 1, 2, \ldots),$$

as proved just above. Therefore φ is by definition integrable by σ in U_P, and

$$\mu(P) = \int_P \varphi(\xi)\sigma(d\xi).$$

In general, for a totally additive function $\mu(P)$ $(P \in \mathfrak{R})$, both $\mathfrak{R}\mu(P)$ and $\mathfrak{I}\mu(P)$ $(P \in \mathfrak{R})$ are obviously totally additive. Therefore, putting

$$\psi_1(\xi) = \lim_{P \to \xi} \frac{\mathfrak{R}\mu(P)}{\sigma(P)}, \quad \psi_2(\xi) = \lim_{P \to \xi} \frac{\mathfrak{I}\mu(P)}{\sigma(P)} \quad (\xi \in U_{P_0}),$$

we obtain two continuous functions ψ_1 and ψ_2 on U_{P_0}, and

$$\mathfrak{R}\mu(P) = \int_P \psi_1(\xi)\sigma(d\xi), \quad \mathfrak{I}\mu(P) = \int_P \psi_2(\xi)\sigma(d\xi)$$

for every $\mathfrak{R} \ni P \leqq P_0$, as proved just above. Thus, putting

$$\varphi(\xi) = \psi_1(\xi) + i\,\psi_2(\xi),$$

we obtain a continuous function φ on U_{P_0}, and we conclude easily by the formula §16(2)

$$\mu(P) = \int \varphi(\xi)\sigma(d\xi) \qquad \text{for every } \mathfrak{R} \ni P \leqq P_0.$$

Theorem 18.3. Let φ be integrable by a totally additive function $\mu(P)$ $(P \in \mathfrak{R})$ in U_{P_0}, and a positive totally additive function $\sigma(P)$ $(P \in \mathfrak{R})$ proper in U_{P_0}. For a point $\xi_0 \in U_{P_0}$, if φ is continuous and finite at ξ_0, and $\lim_{P \to \xi_0} \frac{\mu(P)}{\sigma(P)}$ is finite, then, putting

$$\mu_1(P) = \int_P \varphi(\xi)\mu(d\xi) \qquad (\mathfrak{R} \ni P \leqq P_0),$$

we have

$$\lim_{P \to \xi_0} \frac{\mu_1(P)}{\sigma(P)} = \varphi(\xi_0) \lim_{P \to \xi_0} \frac{\mu(P)}{\sigma(P)}$$

Proof. Since φ is finite continuous at ξ_0 by assumption, for any $\varepsilon > 0$ we can find $Q_1 \in \mathfrak{R}$ such that $\xi_0 \in U_{Q_1}$ and

$$|\varphi(\xi) - \varphi(\xi_0)| \leqq \varepsilon \qquad \text{for every } \xi \in U_{Q_1 P_0}.$$

Since the limit $\lim_{P \to \xi_0} \frac{\mu(P)}{\sigma(P)}$ is finite by assumption, we can find $\gamma > 0$ and $Q_2 \in \mathfrak{R}$ such that $\xi_0 \in U_{Q_2}$ and

$$\left| \frac{\mu(P)}{\sigma(P)} \right| \leqq \gamma \qquad \text{for } \xi_0 \ni P \leqq Q_2.$$

Then $\gamma\sigma(P)$ $(P \in \mathfrak{R})$ is a majorant of $\mu(Q_2 P)$ $(P \in \mathfrak{R})$, and for $\xi_0 \ni P \leqq Q_1 Q_2 P_0$ we have by the formula §16(6)

$$|\mu_1(P) - \varphi(\xi_0)\mu(P)| = \left| \int_P \{\varphi(\xi) - \varphi(\xi_0)\}\mu(d\xi) \right|$$

$$\leqq \int_P \varepsilon\gamma\sigma(d\xi) = \varepsilon\gamma\sigma(P).$$

Thus we can conclude by definition

$$\lim_{P \to \xi_0} \frac{\mu_1(P)}{\sigma(P)} = \varphi(\xi_0) \lim_{P \to \xi_0} \frac{\mu(P)}{\sigma(P)}$$

For every $a \in R$, $\|Pa\|^2$ $(P \in \mathfrak{R})$ is obviously positive and totally additive. Furthermore, it is proper in the cover $C_{[a]}$, because $\|Pa\| = 0$ implies by the formula §14(16')

$$P C_{[a]} = C_{[Pa]} = 0.$$

By virtue of Theorem 18.2, for every $b \in R$, putting

$$\varphi(\xi) = \lim_{P \to \xi} \frac{(Pb, a)}{\|Pa\|^2} \qquad \text{for } \xi \in U_{C_{[a]}},$$

we obtain a continuous function φ on $U_{C_{[a]}}$, and

$$(P\mathfrak{k}, a) = \int_P \varphi(\mathfrak{x}) \| d_{\mathfrak{x}} a \|^2 \qquad \text{for } \mathfrak{R} \ni P \leqq C_{[a]}.$$

For every sequence $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} C_{[a]}$ such that φ is bounded in $\mathcal{U}_{P_\nu}$ for every $\nu = 1, 2, \ldots$, putting

$$\mathfrak{k}_\nu = \int_{P_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} a \qquad (\nu = 1, 2, \ldots),$$

we have by §17(4) for every $P \in \mathfrak{R}$

$$P\mathfrak{k}_\nu = \int_{PP_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} a,$$

and hence by §17(6)

$$(P\mathfrak{k}_\nu, a) = \int_{PP_\nu} \varphi(\mathfrak{x}) \| d_{\mathfrak{x}} a \|^2 = (PP_\nu \mathfrak{k}, a),$$

that is, $(\mathfrak{k}_\nu, Pa) = (P_\nu \mathfrak{k}, Pa)$ for every $P \in \mathfrak{R}$. This relation yields by the formula §14(9)

$$[a]\mathfrak{k}_\nu = [a]P_\nu \mathfrak{k} = P_\nu [a]\mathfrak{k}.$$

On the other hand we have by Theorem 17.11 and the formula §14(3)

$$[a]\mathfrak{k}_\nu = \int_{P_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} [a]a = \int_{C_{\mathfrak{R}} P_\nu} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} a = \mathfrak{k}_\nu.$$

Thus we obtain $\mathfrak{k}_\nu = P_\nu [a]\mathfrak{k}$ for every $\nu = 1, 2, \ldots$, and hence

$$\lim_{\nu \to \infty} \mathfrak{k}_\nu = C_{[a]}[a]\mathfrak{k} = [a]\mathfrak{k},$$

that is,

$$[a]\mathfrak{k} = \int_{C_{[a]}} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} a.$$

The value of this function $\varphi(\mathfrak{x})$ is called the relative spectrum of an element $\mathfrak{k}$ by a at $\mathfrak{x} \in \mathcal{U}_{C_{[a]}}$, and denoted by $\left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right)$, i.e.

$$\left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right) = \lim_{P \to \mathfrak{x}} \frac{(P\mathfrak{k}, a)}{\| Pa \|^2} \qquad (\mathfrak{x} \in \mathcal{U}_{C_{[a]}}).$$

With this definition we can state therefore

Theorem 18.4. *The relative spectrum* $\left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right)$ *is a continuous function on* $\mathcal{U}_{C_{[a]}}$, *integrable by* a *in* $\mathcal{U}_{C_{[a]}}$, *and*

$$[a]\mathfrak{k} = \int_{C_{[a]}} \left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right) d_{\mathfrak{x}} a.$$

Theorem 18.5. $\mathfrak{k} = \int_{C_{[a]}} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} c$ *implies*

$$\left(\frac{\mathfrak{k}}{c}, \mathfrak{x}\right) = \varphi(\mathfrak{x}) \left(\frac{c}{a}, \mathfrak{x}\right)$$

if both $\varphi(\mathfrak{x})$ *and* $\left(\frac{c}{a}, \mathfrak{x}\right)$ *are finite continuous at* $\mathfrak{x} \in \mathcal{U}_{C_{[a]}}$.

Proof. Recalling Theorem 17.4, we have by assumption

$$P\mathfrak{k} = \int_{P C_{[a]}} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} C = \int_{P} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} C$$

for $\mathfrak{R} \ni P \leqq C_{[a]}$. Thus we obtain by Theorem 17.10

$$(P\mathfrak{k}, a) = \int_{P} \varphi(\mathfrak{x})\, (d_{\mathfrak{x}} C, a) \qquad \text{for } \mathfrak{R} \ni P \leqq C_{[a]},$$

and hence by Theorem 18.3

$$\left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right) = \lim_{P \to \mathfrak{x}} \frac{(P\mathfrak{k}, a)}{\|Pa\|^2} = \varphi(\mathfrak{x}) \lim_{P \to \mathfrak{x}} \frac{(Pc, a)}{\|Pa\|^2} = \varphi(\mathfrak{x}) \left(\frac{c}{a}, \mathfrak{x}\right)$$

if both $\varphi(\mathfrak{x})$ and $\left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right)$ are finite continuous at $\mathfrak{x} \in U_{C_{[a]}}$.

§19 Properties of Relative Spectra

The relative spectrum $\left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right)$ is defined only for $\mathfrak{x} \in U_{C_{[a]}}$ as

$$(1) \qquad \left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right) = \lim_{P \to \mathfrak{x}} \frac{(P\mathfrak{k}, a)}{\|Pa\|^2}.$$

With this definition, we have obviously

$$(2) \qquad \left(\frac{a}{a}, \mathfrak{x}\right) = 1,$$

$$(3) \qquad \left(\frac{\mathfrak{k}}{\alpha a}, \mathfrak{x}\right) = \frac{1}{\alpha} \left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right) \qquad \text{for } \alpha \neq 0,$$

$$(4) \qquad \left(\frac{\alpha\mathfrak{k}}{a}, \mathfrak{x}\right) = \alpha \left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right) \qquad \text{for every complex number } \alpha,$$

adopting the convention $0 \cdot \infty = 0$.

Recalling Theorem 18.1, we obtain immediately by (1)

$$(5) \qquad \left(\frac{\mathfrak{k} + c}{a}, \mathfrak{x}\right) = \left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right) + \left(\frac{c}{a}, \mathfrak{x}\right)$$

if the right side has any sense.

We have obviously by the definition (1)

$$(6) \qquad \left(\frac{P\mathfrak{k}}{a}, \mathfrak{x}\right) = \left(\frac{\mathfrak{k}}{Pa}, \mathfrak{x}\right) = \left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right) \qquad \text{for } \mathfrak{x} \in U_P.$$

By virtue of the formula §14(3), we have for every $P \in \mathfrak{R}$

$$(P[a]\mathfrak{k}, a) = (P\mathfrak{k}, [a]a) = (P\mathfrak{k}, a).$$

Thus we obtain by the definition (1)

$$(7) \qquad \left(\frac{[a]\mathfrak{k}}{a}, \mathfrak{x}\right) = \left(\frac{\mathfrak{k}}{a}, \mathfrak{x}\right).$$

If $\mathfrak{x} \overline{\in} U_P$, then we have $\mathfrak{x} \in U_P' = U_{C_{\mathfrak{R}} - P}$, and hence by (6) and (4)

$$\left(\frac{P\mathfrak{k}}{a}, \mathfrak{x}\right) = \left(\frac{(C_{\mathfrak{R}} - P)P\mathfrak{k}}{a}, \mathfrak{x}\right) = 0.$$

Thus we have

$$(8) \qquad \left(\frac{P\ell}{a}, \mathfrak{P}\right) = 0 \qquad \text{for } \mathfrak{P} \in \mathfrak{V}_P.$$

We also can prove likewise by (7) and (4)

$$(9) \qquad \left(\frac{(1-[a])\ell}{a}, \mathfrak{P}\right) = 0.$$

Recalling Theorem 17.4, we obtain by Theorem 18.4

$$(10) \qquad P[a]\ell = \int_{PC_{[a]}} \left(\frac{\ell}{a}, \mathfrak{P}\right) d_{\mathfrak{P}} a \qquad \text{for every } P \in \mathcal{R}.$$

Thus we have by Theorem 17.3

$$(11) \qquad \|P[a]\ell\|^2 = \int_{PC_{[a]}} \left|\left(\frac{\ell}{a}, \mathfrak{P}\right)\right|^2 \|d_{\mathfrak{P}} a\|^2 \qquad (P \in \mathcal{R}).$$

This relation yields by Theorem 18.3

$$(12) \qquad \lim_{P \to \mathfrak{P}} \frac{\|P[a]\ell\|}{\|Pa\|} = \left|\left(\frac{\ell}{a}, \mathfrak{P}\right)\right|$$

if the right side is finite. However, both $\lim_{P \to \mathfrak{P}} \frac{\|P[a]\ell\|}{\|Pa\|}$ and and $\left|\left(\frac{\ell}{a}, \mathfrak{P}\right)\right|$ are continuous over $\mathfrak{V}_{C_{[a]}}$, and $\left|\left(\frac{\ell}{a}, \mathfrak{P}\right)\right|$ is finite in some open set being dense in $\mathfrak{V}_{C_{[a]}}$. Thus (12) holds for every $\mathfrak{P} \in \mathfrak{V}_{C_{[a]}}$.

Recalling Theorem 17.10, we conclude from (10)

$$(13) \qquad (P[a]\ell, c) = \int_{PC_{[a]}} \left(\frac{\ell}{a}, \mathfrak{P}\right) (d_{\mathfrak{P}} a, c) \qquad (P \in \mathcal{R}).$$

Consequently we obtain by Theorem 18.3

$$\lim_{P \to \mathfrak{P}} \frac{(P[a]\ell, c)}{\|Pc\|^2} = \left(\frac{\ell}{a}, \mathfrak{P}\right) \lim_{P \to \mathfrak{P}} \frac{(Pa, c)}{\|Pc\|^2}$$

and hence

$$(14) \qquad \left(\frac{[a]\ell}{c}, \mathfrak{P}\right) = \left(\frac{\ell}{a}, \mathfrak{P}\right)\left(\frac{a}{c}, \mathfrak{P}\right) \qquad \text{for } \mathfrak{P} \in \mathfrak{V}_{C_{[a]}C_{[c]}}.$$

if both $\left(\frac{\ell}{a}, \mathfrak{P}\right)$ and $\left(\frac{a}{c}, \mathfrak{P}\right)$ are finite.

Putting $c = [a]\ell$ in (14), we obtain by (2)

$$(15) \qquad \left(\frac{a}{[a]\ell}, \mathfrak{P}\right) = \frac{1}{\left(\frac{\ell}{a}, \mathfrak{P}\right)} \qquad \text{for } \mathfrak{P} \in \mathfrak{V}_{C_{[[a]\ell]}},$$

because both sides are continuous in $\mathfrak{V}_{C_{[[a]\ell]}}$ and finite in some open set being dense in $\mathfrak{V}_{C_{[[a]\ell]}}$.

Recalling Theorem 17.9, we obtain further by (10)

$$(16) \qquad (P[a]\ell, c) = \int_{PC_{[a]}} \left(\frac{\ell}{a}, \mathfrak{P}\right) \overline{\left(\frac{c}{a}, \mathfrak{P}\right)} \|d_{\mathfrak{P}} a\|^2 \qquad (P \in \mathcal{R}).$$

For a continuous function φ on a neighbourhood $\mathfrak{V}_{P_0}$, since

$|\varphi(\xi)|$ also is continuous in U_{P_0}, the point set

$$\{\xi : |\varphi(\xi)| > 0\}$$

is open. Thus there exists by Theorems 15.6 and 15.9 uniquely $P \in \mathcal{R}$ for which

$$U_P = \{\xi : P(\xi) \neq 0\}.$$

This projection operator P is called the cover of φ and denoted by C_φ. Concerning covers of functions, we have

Theorem 19.1. *For a continuous function φ on U_{P_0}, if*

$$\int_P \varphi(\xi)\, d_\xi a = 0, \qquad \mathcal{R} \ni P \leq P_0,$$

then we have $P C_\varphi C_{[a]} = 0$ for the cover C_φ of φ.

Proof. By virtue of Theorems 15.6 and 15.9, we can find a sequence $P_\nu \in \mathcal{R}$ $(\nu = 1, 2, \ldots)$ such that

$$U_{P_\nu} = \{\xi : |\varphi(\xi)| > \tfrac{1}{\nu}\}^- \qquad (\nu = 1, 2, \ldots).$$

Then we have by Theorem 15.7 $P_\nu \uparrow_{\nu=1}^{\infty} C_\varphi$, because we have

$$\{\xi : |\varphi(\xi)| > 0\} = \sum_{\nu=1}^{\infty} \{\xi : |\varphi(\xi)| > \tfrac{1}{\nu}\},$$

and hence

$$\{\xi : |\varphi(\xi)| > 0\}^- = \Big(\sum_{\nu=1}^{\infty} \{\xi : |\varphi(\xi)| > \tfrac{1}{\nu}\}^-\Big)^-.$$

Furthermore we have by Theorem 17.4 for every $\nu = 1, 2, \ldots$

$$\int_{PP_\nu} \varphi(\xi)\, d_\xi a = P_\nu \int_P \varphi(\xi)\, d_\xi a = 0,$$

and hence by Theorem 17.3

$$\int_{PP_\nu} |\varphi(\xi)|^2 \| d_\xi a \|^2 = 0.$$

On the other hand we have by the formula §16(6)

$$\int_{PP_\nu} |\varphi(\xi)|^2 \| d_\xi a \|^2 \geq \int_{PP_\nu} \frac{1}{\nu^2} \| d_\xi a \|^2 = \frac{1}{\nu^2} \| P P_\nu a \|^2.$$

Consequently we obtain $\| P P_\nu a \| = 0$ for every $\nu = 1, 2, \ldots$. This relation yields $P C_\varphi a = 0$, and hence $P C_\varphi C_{[a]} = 0$ by the formula §14(19).

Theorem 19.2. *The cover of the relative spectrum $(\frac{b}{a}, \xi)$ coincides with the cover of $[a]b$, that is,*

$$\{\xi : (\tfrac{b}{a}, \xi) \neq 0\}^- = U_{C_{[a]b}}.$$

Proof. Denoting by C_0 the cover of $(\frac{\ell}{a}, \xi)$ we have obviously $C_0 \leqq C_{[a]}$, and by the formula (10)

$$[a]\ell = \int_{C_{[a]}} (\frac{\ell}{a}, \xi)\, d_\xi a = \int_{C_0} (\frac{\ell}{a}, \xi)\, d_\xi a.$$

Thus we have $C_0[a]\ell = [a]\ell$ by Theorem 17.5, and hence we obtain $C_0 \geq C_{[[a]\ell]}$ by the formula §14(18). On the other hand, since $(1 - C_{[[a]\ell]})[a]\ell = 0$, we have by Theorem 17.5

$$\int_{C_{[a]}(1 - C_{[[a]\ell]})} (\frac{\ell}{a}, \xi)\, d_\xi a = 0.$$

Thus we obtain by Theorem 19.1

$$C_0\, C_{[a]}(1 - C_{[[a]\ell]}) = 0,$$

and hence $C_0 = C_0\, C_{[[a]\ell]} \leqq C_{[[a]\ell]}$.

Theorem 19.3. $C_{[a]} = C_{[\ell]}$, $[a] \geqq [\ell]$ implies $[a] = [\ell]$.

Proof. Recalling Theorem 17.11, we obtain by (10)

$$\int_{C_{[a]}} (\frac{\ell}{a}, \xi)\, d_\xi ([a] - [\ell])a = ([a] - [\ell])\ell = 0,$$

because we conclude $[a]\ell = C_\mathfrak{N}\ell$ by the relation §14(10) from the assumption: $[a] \geqq [\ell]$. Since $C_{[\ell]}$ is by Theorem 19.2 the cover of $(\frac{\ell}{a}, \xi)$, we obtain hence by Theorem 19.1

$$C_{[a]}\, C_{[\ell]}\, C_{[([a] - [\ell])a]} = 0.$$

Since $C_{[a]} = C_{[\ell]}$ by assumption, we obtain consequently

$$[a][([a] - [\ell])a] = 0.$$

This relation yields by the formula §14(8)

$$[a]([a] - [\ell])a = 0,$$

and hence $[\ell]a = [a]a = C_\mathfrak{N} a$. Thus we obtain $[a] \leqq [\ell]$ by §14(10). Consequently we have $[a] = [\ell]$.

Theorem 19.4. $[a] \geqq [\ell]$ implies $[\ell] = C_{[\ell]}[a]$.

Proof. Recalling §14(6'), we obtain by assumption

$$[C_{[\ell]}a] = C_{[\ell]}[a] \geqq [\ell].$$

Furthermore we have by the formulas (16') and (12) in §14

$$C_{[C_{[\ell]}a]} = C_{[a]}\, C_{[\ell]} = C_{[\ell]}.$$

Thus we conclude by Theorem 19.3

$$[\ell] = [C_{[\ell]}a] = C_{[\ell]}[a].$$

If $C_{[a]}\, C_{[\ell]} = 0$, then we have by Theorem 14.5

$$[a+\ell] = [a] \vee [\ell] = [a] + [\ell],$$

because $[a][\ell] = 0$ by assumption. Conversely if

$$[a+\ell] = [a] + [\ell],$$

then we have by Theorem 12.4

$$[a+\ell] \geqq [a], \quad [a+\ell] \geqq [\ell], \quad [a][\ell] = 0,$$

and hence by Theorem 19.4

$$C_{[a]} C_{[\ell]} [a+\ell] = 0.$$

This relation yields by the formulas (12) and (14) in §14

$$C_{[a]} C_{[\ell]} = C_{[a]} C_{[\ell]} C_{[a+\ell]} = 0.$$

Therefore we can state

Theorem 19.5. $[a+\ell] = [a] + [\ell]$ *if and only if*

$$C_{[a]} C_{[\ell]} = 0.$$

Theorem 19.6. *If* $[a] \geqq [\ell]$, *then we have*

$$[a] - [\ell] = [(1 - C_{[\ell]})a].$$

Proof. By virtue of the formula §14(16'), we have

$$C_{[(1-C_{[\ell]})a]} C_{(C_{[\ell]}a)} = (1 - C_{[\ell]}) C_{[\ell]} C_{[a]} = 0,$$

and hence by Theorem 19.5

$$[a] = [(1 - C_{[\ell]})a] + [C_{[\ell]}a].$$

If $[a] \geqq [\ell]$, then we have by Theorem 19.4

$$[\ell] = C_{[\ell]}[a] = [C_{[\ell]}a].$$

Thus we obtain our assertion.

§20 Properties of Rings

Let $\mathfrak{R}$ be a ring of projection operators in a Hilbert space $\mathfrak{H}$, and $\overline{\mathfrak{R}}$ the commutor of $\mathfrak{R}$. For the commutor $\overline{\overline{\mathfrak{R}}}$ of $\overline{\mathfrak{R}}$, we have obviously

$$\mathfrak{R} \subset \overline{\overline{\mathfrak{R}}} \subset \overline{\mathfrak{R}},$$

because $\mathfrak{R} \subset \overline{\mathfrak{R}}$. Thus we see easily that $\overline{\overline{\mathfrak{R}}}$ also is a ring, and furthermore closed by Theorem 13.7. For the cover $C_{\mathfrak{R}}$ of $\mathfrak{R}$, we see easily that $C_{\mathfrak{R}} \in \overline{\overline{\mathfrak{R}}}$ and $C_{\mathfrak{R}}\overline{\overline{\mathfrak{R}}}$ is a closed subring of $\overline{\overline{\mathfrak{R}}}$ including $\mathfrak{R}$. Thus we can conclude that for any ring $\mathfrak{R}$ there exists the least closed ring including $\mathfrak{R}$. Furthermore $C_{\mathfrak{R}}\overline{\overline{\mathfrak{R}}}$ is the least closed ring including $\mathfrak{R}$, that is, we have

Theorem 20.1. *For a closed ring* $\mathfrak{R}$, *we have* $\mathfrak{R} = C_{\mathfrak{R}}\overline{\overline{\mathfrak{R}}}$, *that*

is, if a projection operator Q is commutative to the commutor $\overline{\mathfrak{R}}$ of $\mathfrak{R}$, then we have $C_{\mathfrak{R}}Q \in \mathfrak{R}$ for the cover $C_{\mathfrak{R}}$ of $\mathfrak{R}$.

This Theorem 20.1 is concluded at once from

Theorem 20.2. Let $\mathfrak{R}$ be a closed ring. For a projection operator Q, if $Q[x] = [x]Q$ for every $x \in R$, then $C_{\mathfrak{R}}Q \in \mathfrak{R}$.

Proof. By virtue of Theorem 14.1, we have for every $P \in \mathfrak{R}$

$$P = \bigcup_{x \in R} [Px],$$

and hence we conclude by Theorem 13.6 from assumption that $QP = PQ$ for every $P \in \mathfrak{R}$, that is, $Q \in \overline{\mathfrak{R}}$. Thus we have by §14(1)

$$[Qx] \leqq C_{\mathfrak{R}}Q \qquad \text{for every } x \in R.$$

Furthermore we have

$$C_{[Qx]} \leqq C_{\mathfrak{R}}Q \qquad \text{for every } x \in R.$$

Because, if we can find $a, \ell \in R$ such that

$$\ell = C_{[Qa]}(1-Q)\ell \neq 0,$$

then we have by §14(6) and the assumption: $Q[x] = [x]Q$ for $x \in R$

$$Q[Qa+\ell] = [Q(Qa+\ell)] = [Qa],$$
$$(1-Q)[Qa+\ell] = [(1-Q)\ell] = [\ell],$$

and hence

$$[Qa+\ell] = [Qa] + [\ell].$$

Thus we obtain by Theorem 19.5

$$C_{[Qa]}\, C_{[\ell]} = 0.$$

Since $C_{[Qa]}\ell = \ell$, we obtain hence by §14(16')

$$C_{[\ell]} = C_{[C_{[Qa]}\ell]} = C_{[Qa]}C_{[\ell]} = 0,$$

contradicting the assumption: $\ell \neq 0$. Therefore we have

$$C_{[Qx]}(1-Q) = 0 \qquad \text{for every } x \in R,$$

and hence $C_{[Qx]} = C_{[Qx]}Q \leqq C_{\mathfrak{R}}Q$.

Putting $P_0 = \bigcup_{x \in R} C_{[Qx]}$ we have $P_0 \in \mathfrak{R}$, because $\mathfrak{R}$ is closed by assumption. And we have $P_0 \leqq C_{\mathfrak{R}}Q$, as proved just above. For every $x \in (C_{\mathfrak{R}}Q - P_0)R$, we have naturally $Qx = x$, $P_0 x = 0$, and hence by §14(16')

$$C_{[x]} = C_{[Qx]} = C_{[Qx]}P_0 = C_{[x]}P_0 = C_{[P_0 x]} = 0,$$

because $C_{[Qx]} \leqq P_0$. Therefore we have $C_{\mathfrak{R}}Q - P_0 = 0$, and hence $C_{\mathfrak{R}}Q = P_0 \in \mathfrak{R}$.

For a commutative collection of projection operators $\mathfrak{P}$, we also have obviously $\mathfrak{P} \subset \overline{\overline{\mathfrak{P}}} \subset \overline{\mathfrak{P}}$. Thus, putting $P_0 = \bigcup_{P \in \mathfrak{P}} P$, we see by Theorem 20.1 that $P_0 \overline{\overline{\mathfrak{P}}}$ is the least closed ring including $\mathfrak{P}$, which will be called the closed ring <u>generated</u> by $\mathfrak{P}$.

For every $Q \in \overline{\mathfrak{R}}$, we see easily that $Q\mathfrak{R}$ is a ring with cover $Q\mathcal{C}_{\mathfrak{R}}$, and if $\mathfrak{R}$ is closed, then $Q\mathfrak{R}$ also is closed. Between the projectors $[x]_{Q\mathfrak{R}}$ by $Q\mathfrak{R}$ and $[x]_{\mathfrak{R}}$ by $\mathfrak{R}$, we have obviously by definition

$$(1) \qquad [x]_{Q\mathfrak{R}} = [Qx]_{\mathfrak{R}}.$$

Between the covers $\mathcal{C}_{\rho}^{Q\mathfrak{R}}$ by $Q\mathfrak{R}$ and $\mathcal{C}_{\rho}^{\mathfrak{R}}$ by $\mathfrak{R}$ we have obviously by definition

$$(2) \qquad \mathcal{C}_{\rho}^{Q\mathfrak{R}} = Q\,\mathcal{C}_{\rho}^{\mathfrak{R}} \quad \text{for } P \leqq Q\mathcal{C}_{\mathfrak{R}}.$$

Putting $Q = [a]_{\mathfrak{R}}$ in (1), we obtain by Theorem 19.4

$$(3) \qquad [x]_{[a]_{\mathfrak{R}}\mathfrak{R}} = [a]_{\mathfrak{R}}\,\mathcal{C}_{[[a]_{\mathfrak{R}}x]_{\mathfrak{R}}} \in [a]_{\mathfrak{R}}\,\mathfrak{R}.$$

A ring $\mathfrak{R}$ is said to be <u>simple</u>, if $\mathfrak{R} = \mathcal{C}_{\mathfrak{R}}\overline{\mathfrak{R}}$. Thus every simple ring is closed.

<u>Theorem</u> 20.3. <u>A closed ring $\mathfrak{R}$ is simple, if and only if we have $[x] \in \mathfrak{R}$ for every $x \in R$.</u>

<u>Proof</u>. Since $[x] \in \overline{\mathfrak{R}}$ by §14(5), if $\mathfrak{R}$ is simple, then we have $[x] = \mathcal{C}_{\mathfrak{R}}[x] \in \mathfrak{R}$. Conversely if $[x] \in \mathfrak{R}$ for every $x \in R$, then for any $Q \in \overline{\mathfrak{R}}$ we have by Theorem 14.1

$$\mathcal{C}_{\mathfrak{R}}Q = \bigcup_{x \in R} [Qx] \in \mathfrak{R},$$

because $\mathfrak{R}$ is closed by assumption.

<u>Theorem</u> 20.4. <u>In order that a closed ring $\mathfrak{R}$ be simple, it is necessary and sufficient that for some $Q \in \overline{\mathfrak{R}}$, if $QP = 0$, $P \in \mathfrak{R}$ implies $P = 0$, then $Q \geqq \mathcal{C}_{\mathfrak{R}}$.</u>

<u>Proof</u>. We suppose firstly that $\mathfrak{R}$ is simple. For some $Q \in \overline{\mathfrak{R}}$, if we have not $Q \geqq \mathcal{C}_{\mathfrak{R}}$, then there exists $a \in R$ such that

$$a = \mathcal{C}_{\mathfrak{R}}(1-Q)a \neq 0.$$

For such a we have by §14(1)

$$[a] = \mathcal{C}_{\mathfrak{R}}(1-Q)[a],$$

and hence $Q[a] = 0$, $[a] \in \mathfrak{R}$, but $[a] \neq 0$.

If $\mathfrak{R}$ is not simple, then there exists by definition $Q \in \overline{\mathfrak{R}}$ such that $C_{\mathfrak{R}} \geq Q \overline{\in} \mathfrak{R}$. For such Q, putting $Q_0 = Q + (C_{\mathfrak{R}} - C_Q)$, we have by §14(15) and §14(11)

$$C_{Q_0} = C_Q + (C_{\mathfrak{R}} - C_Q) = C_{\mathfrak{R}}.$$

Thus $Q_0 P = 0$, $P \in \mathfrak{R}$ implies $P = 0$ by §14(14), but

$$C_{\mathfrak{R}} - Q_0 = C_Q - Q \neq 0$$

because $Q \overline{\in} \mathfrak{R}$.

Recalling the formula (3), we obtain immediately by Theorem 20.3

Theorem 20.5. For any closed ring $\mathfrak{R}$, $[a]\mathfrak{R}$ is a simple ring for every $a \in R$.

Theorem 20.6. If a closed ring $\mathfrak{R}$ is simple, then $Q\mathfrak{R}$ also is a simple ring for every $Q \in \overline{\mathfrak{R}}$.

Proof. If a projection operator P_0 is commutative to $Q\mathfrak{R}$, and $P_0(QP) = 0$, $P \in \mathfrak{R}$ implies $QP = 0$, then, putting

$$P_1 = (QC_{\mathfrak{R}})P_0 + (C_{\mathfrak{R}} - QC_{\mathfrak{R}}),$$

we see easily that $P_1 \in \overline{\mathfrak{R}}$, and $P_1 P = 0$, $P \in \mathfrak{R}$ implies $P = 0$. Thus we have $P_1 \geq C_{\mathfrak{R}}$ by Theorem 20.4, and hence $(QC_{\mathfrak{R}})P_0 = QC_{\mathfrak{R}}$. Therefore $Q\mathfrak{R}$ is simple by Theorem 20.4.

A ring of projection operators $\mathfrak{R}$ is said to be an axis, if $\mathfrak{R}$ is simple and $1 \in \mathfrak{R}$, that is, if $\mathfrak{R} = \overline{\mathfrak{R}}$. Every simple ring $\mathfrak{R}$ may be obviously consided as an axis in the subspace $C_{\mathfrak{R}} R$.

By virtue of Maximal Theorem, we have obviously by definition

Theorem 20.7. For any commutative collection of projection operators $\mathfrak{P}$ we can find an axis including $\mathfrak{P}$.

Let $\mathfrak{R}$ be a closed ring containing 1, and $\mathfrak{T}$ a closed subring containing 1, that is, $1 \in \mathfrak{T} \subset \mathfrak{R}$, and both $\mathfrak{R}$ and $\mathfrak{T}$ are closed rings of projection operators For each maximal ideal $\mathfrak{P}$ in $\mathfrak{R}$, putting

$$\mathfrak{P}^{\mathfrak{T}} = \mathfrak{T}\mathfrak{P}$$

we obtain obviously an ideal $\mathfrak{P}^{\mathfrak{T}}$ in $\mathfrak{T}$. Furthermore $\mathfrak{P}^{\mathfrak{T}}$ is maximal. In fact, if $\mathfrak{T} \ni P \overline{\in} \mathfrak{P}^{\mathfrak{T}}$, then we also have $P \overline{\in} \mathfrak{P}$, and hence $1 - P \in \mathfrak{P}$. This relation yields $1 - P \in \mathfrak{P}^{\mathfrak{T}}$, while $P(1-P) = 0$. Thus $\mathfrak{P}^{\mathfrak{T}}$ is by Theorem 15.2 a maximal ideal in $\mathfrak{T}$.

Conversely, for each maximal ideal $\mathfrak{q}$ in $\mathfrak{r}$, we can find by Theorem 15.1 a maximal ideal $\mathfrak{P}_0$ in $\mathfrak{R}$, such that $\mathfrak{P}_0 \supset \mathfrak{q}$. For such $\mathfrak{P}_0$ we have obviously

$$\mathfrak{P}_0^{\mathfrak{r}} = \mathfrak{r}\mathfrak{P}_0 \supset \mathfrak{q}$$

and $\mathfrak{P}_0^{\mathfrak{r}}$ also is a maximal ideal in $\mathfrak{r}$. Since $\mathfrak{q}$ is maximal by assumption, we obtain hence $\mathfrak{q} = \mathfrak{P}_0^{\mathfrak{r}}$. Therefore, putting

$$(4) \qquad \mathfrak{P}^{\mathfrak{r}} = \mathfrak{r}\mathfrak{P} \in \mathfrak{L}_{\mathfrak{r}} \qquad (\mathfrak{P} \in \mathfrak{L}_{\mathfrak{R}}),$$

we obtain a mapping of the proper space $\mathfrak{L}_{\mathfrak{R}}$ of $\mathfrak{R}$ onto the proper space $\mathfrak{L}_{\mathfrak{r}}$ of $\mathfrak{r}$. Furthermore, between the neighbourhoods $\mathfrak{U}_P^{\mathfrak{r}}$ of $\mathfrak{L}_{\mathfrak{r}}$ and $\mathfrak{U}_P^{\mathfrak{R}}$ of $\mathfrak{L}_{\mathfrak{R}}$, we have

$$(5) \qquad \mathfrak{U}_P^{\mathfrak{R}} = \{ \mathfrak{P} : \mathfrak{r}\mathfrak{P} \in \mathfrak{U}_P^{\mathfrak{r}},\ \mathfrak{P} \in \mathfrak{L}_{\mathfrak{R}} \} \qquad (P \in \mathfrak{r}),$$

because $\mathfrak{U}_P^{\mathfrak{R}} \ni \mathfrak{P}$ implies $P \in \mathfrak{P}$ and hence $P \in \mathfrak{r}\mathfrak{P}$, that is, $\mathfrak{r}\mathfrak{P} \in \mathfrak{U}_P^{\mathfrak{r}}$; and conversely $\mathfrak{r}\mathfrak{P} \in \mathfrak{U}_P^{\mathfrak{r}}$ implies $P \in \mathfrak{P}$, that is, $\mathfrak{P} \in \mathfrak{U}_P^{\mathfrak{R}}$.

For every $Q \in \mathfrak{R}$, we shall denote by $\mathfrak{C}_Q^{\mathfrak{r}}$ the cover of Q by $\mathfrak{r}$. With this notation we have

$$(6) \qquad \mathfrak{U}_{\mathfrak{C}_Q^{\mathfrak{r}}}^{\mathfrak{r}} = \{ \mathfrak{r}\mathfrak{P} : \mathfrak{P} \in \mathfrak{U}_Q^{\mathfrak{R}} \} \qquad (Q \in \mathfrak{R}).$$

Because $\mathfrak{P} \in \mathfrak{U}_Q^{\mathfrak{R}}$ implies $\mathfrak{P} \ni Q \leqq \mathfrak{C}_Q^{\mathfrak{r}} \in \mathfrak{r}$ and hence $\mathfrak{r}\mathfrak{P} \ni \mathfrak{C}_Q^{\mathfrak{r}}$. Conversely, for any $\mathfrak{q} \in \mathfrak{U}_{\mathfrak{C}_Q^{\mathfrak{r}}}^{\mathfrak{r}}$, we have $\mathfrak{C}_Q^{\mathfrak{r}} \in \mathfrak{q}$ and hence

$$\mathfrak{C}_Q^{\mathfrak{r}} P \neq 0 \qquad \text{for every } P \in \mathfrak{q}.$$

Since $\mathfrak{C}_Q^{\mathfrak{r}} P = \mathfrak{C}_{PQ}^{\mathfrak{r}}$ by the formula §14(16'), we have hence $PQ \neq 0$ for every $P \in \mathfrak{q}$. Thus we can find by Theorem 15.1 $\mathfrak{P} \in \mathfrak{L}_{\mathfrak{R}}$ for which $\mathfrak{P} \ni Q$ and $\mathfrak{P} \supset \mathfrak{q}$; and hence $\mathfrak{P} \in \mathfrak{U}_Q^{\mathfrak{R}}$ and $\mathfrak{r}\mathfrak{P} = \mathfrak{q}$.

Now we can state

Theorem 20.8. *For two closed rings of projection operators*

$$1 \in \mathfrak{r} < \mathfrak{R}$$

the proper space $\mathfrak{L}_{\mathfrak{r}}$ of $\mathfrak{r}$ is a partition space of the proper space $\mathfrak{L}_{\mathfrak{R}}$ of $\mathfrak{R}$ by the mapping (4): for each $P \in \mathfrak{r}$, the inverse image of the neighbourhood $\mathfrak{U}_P^{\mathfrak{r}}$ in $\mathfrak{L}_{\mathfrak{r}}$ is the corresponding neighbourhood $\mathfrak{U}_P^{\mathfrak{R}}$ in $\mathfrak{L}_{\mathfrak{R}}$, and for each $Q \in \mathfrak{R}$, the image of $\mathfrak{U}_Q^{\mathfrak{R}}$ is $\mathfrak{U}_{\mathfrak{C}_Q^{\mathfrak{r}}}^{\mathfrak{r}}$ for the cover $\mathfrak{C}_Q^{\mathfrak{r}}$ of Q by $\mathfrak{r}$.

§21 Ring Convergence

Let $\mathfrak{R}$ be a closed ring of projection operators containing 1. For two elements a, $b \in R$, we shall write $a \prec b$ or $b \succ a$, if

$$\|Pa\| \leqq \|Pb\| \qquad \text{for every } P \in \mathfrak{R}.$$

If we need indicate the ring $\mathfrak{R}$, we shall write $a \prec b\ (\mathfrak{R})$ or $b \succ a\ (\mathfrak{R})$. With this definition we have obviously

(1) $x \prec y$, $y \prec z$ implies $x \prec z$,

(2) $|\alpha| \leqq |\beta|$ implies $\alpha x \prec \beta x$ for every $x \in R$,

(3) $x \prec y$ implies $\alpha x \prec \alpha y$ for every complex number α,

(4) $Qx \prec x$ for every $Q \in \overline{\mathfrak{R}}$,

(5) $x \prec y$ implies $Px \prec Py$ for every $P \in \mathfrak{R}$.

If $x \prec y$, then $Py = 0$; $P \in \mathfrak{R}$ implies $Px = 0$ by definition, and hence $P[y] = 0$ implies $P[x] = 0$ by the formulas (4) and (6') in §14. Thus we obtain by definition

$$\mathfrak{C}_{\mathfrak{R}} - \mathfrak{C}_{[x]} = \bigcup_{P[x]=0,\, P\in\mathfrak{R}} P \geqq \bigcup_{P[y]=0,\, P\in\mathfrak{R}} P = \mathfrak{C}_{\mathfrak{R}} - \mathfrak{C}_{[y]}.$$

Therefore we have

(6) $x \prec y$ implies $\mathfrak{C}_{[x]} \leqq \mathfrak{C}_{[y]}$.

We can conclude easily further by definition

(7) $x_\nu \prec y$ $(\nu = 1, 2, \ldots, \varkappa)$ implies $\sum_{\nu=1}^{\varkappa} \alpha_\nu x_\nu \prec \left(\sum_{\nu=1}^{\varkappa} |\alpha_\nu|\right) y$,

(8) $x_\nu \prec y_\nu$, $\lim_{\nu\to\infty} x_\nu = x$, $\lim_{\nu\to\infty} y_\nu = y$ implies $x \prec y$.

Theorem 21.1. We have $[a]x \prec [a]y$ if and only if

$$\left|\left(\frac{x}{a}, \mathfrak{f}\right)\right| \leqq \left|\left(\frac{y}{a}, \mathfrak{f}\right)\right| \qquad \text{for every } \mathfrak{f} \in \mathfrak{C}_{[a]}.$$

Proof. If $[a]x \prec [a]y$, then we have obviously by §19(12)

$$\left|\left(\frac{x}{a}, \mathfrak{f}\right)\right| = \lim_{P\to\mathfrak{f}} \frac{\|P[a]x\|}{\|Pa\|} \leqq \lim_{P\to\mathfrak{f}} \frac{\|P[a]y\|}{\|Pa\|} = \left|\left(\frac{y}{a}, \mathfrak{f}\right)\right|$$

for every $\mathfrak{f} \in \mathfrak{C}_{[a]}$. Conversely if

$$\left|\left(\frac{x}{a}, \mathfrak{f}\right)\right| \leqq \left|\left(\frac{y}{a}, \mathfrak{f}\right)\right| \qquad \text{for every } \mathfrak{f} \in \mathfrak{C}_{[a]},$$

then we have by §19(11) for every $P \in \mathfrak{R}$

$$\|P[a]x\|^2 = \int_{Pe_{[a]}} \left|\left(\frac{x}{a}, \mathfrak{f}\right)\right|^2 \|d\mathfrak{f}\, a\|^2 \leqq \int_{Pe_{[a]}} \left|\left(\frac{y}{a}, \mathfrak{f}\right)\right|^2 \|d\mathfrak{f}\, a\|^2 = \|P[a]y\|^2.$$

Theorem 21.2. If $a_\nu \prec b_\nu$ $(\nu = 1, 2, \ldots)$, $C_{[b_\nu]} C_{[b_\mu]} = 0$ for $\nu \neq \mu$, and $\sum_{\nu=1}^{\infty} \| b_\nu \|^2 < +\infty$, then we have

$$\sum_{\nu=1}^{\infty} a_\nu \prec \sum_{\nu=1}^{\infty} b_\nu .$$

Proof. Since $a_\nu \prec b_\nu$ by assumption, we have $C_{[a_\nu]} \leqq C_{[b_\nu]}$ by (6), and hence we obtain by assumption

$$C_{[a_\nu]} C_{[a_\mu]} = 0 \qquad \text{for} \quad \nu \neq \mu .$$

Since $a_\nu = C_{[a_\nu]} a_\nu$, $b_\nu = C_{[b_\nu]} b_\nu$ by §14(17), both a_ν and b_ν $(\nu = 1, 2, \ldots)$ are orthogonal sequences, and we have by assumption

$$\sum_{\nu=1}^{\infty} \| a_\nu \|^2 \leqq \sum_{\nu=1}^{\infty} \| b_\nu \|^2 < +\infty .$$

Thus both $\sum_{\nu=1}^{\infty} a_\nu$ and $\sum_{\nu=1}^{\infty} b_\nu$ are convergent. Furthermore we have

$$\| P \sum_{\nu=1}^{\infty} a_\nu \|^2 = \sum_{\nu=1}^{\infty} \| P a_\nu \|^2 \leqq \sum_{\nu=1}^{\infty} \| P b_\nu \|^2 = \| P \sum_{\nu=1}^{\infty} b_\nu \|^2$$

for every $P \in \mathfrak{R}$, and hence $\sum_{\nu=1}^{\infty} a_\nu \prec \sum_{\nu=1}^{\infty} b_\nu$ by definition.

Theorem 21.3. For a sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$, if

$$\sum_{\nu=1}^{\infty} \| a_\nu \| < +\infty ,$$

then we can find $b \in R$ for which $a_\nu \prec b$ for every $\nu = 1, 2, \ldots$.

Proof. By virtue of Theorem 14.6, we can find $C \in K$ for which $C_{[a_\nu]} \leqq C_{[C]}$ for every $\nu = 1, 2, \ldots$. For such C, putting

$$\varphi_\nu(\mathfrak{P}) = \lim_{P \to \mathfrak{P}} \frac{\| P a_\nu \|^2}{\| P C \|^2} \qquad (\mathfrak{P} \in \mathfrak{S}_{C_{[C]}}),$$

we obtain by Theorem 18.2 a continuous function φ_ν on $\mathfrak{S}_{C_{[C]}}$ and

$$\| P a_\nu \|^2 = \int_P \varphi_\nu(\mathfrak{P}) \| d\mathfrak{P}\, C \|^2 \qquad \text{for} \quad \mathfrak{R} \ni P \leqq C_{[C]}.$$

Furthermore we can put by Theorem 17.3

$$b_\nu = \int_{C_{[C]}} \sqrt{\varphi_\nu(\mathfrak{P})}\, d\mathfrak{P}\, C \qquad (\nu = 1, 2, \ldots),$$

and we have for every $P \in \mathfrak{R}$

$$\| P b_\nu \|^2 = \int_{P C_{[C]}} \varphi_\nu(\mathfrak{P}) \| d\mathfrak{P}\, C \|^2 = \| P C_{[C]} a_\nu \|^2 = \| P a_\nu \|^2,$$

because $C_{[a_\nu]} \leqq C_{[C]}$ and hence $C_{[C]} a_\nu = a_\nu$ by §14(18). We have further by assumption

$$\sum_{\nu=1}^{\infty} \| b_\nu \| = \sum_{\nu=1}^{\infty} \| a_\nu \| < +\infty ,$$

and hence, putting $\mathfrak{b} = \sum_{\nu=1}^{\infty} \mathfrak{b}_\nu$, we have by Theorems 17.6 and 17.3

$$\| P a_\nu \|^2 = \int_{P e_{[c]}} \varphi_\nu(\xi) \| d_\xi e \, c \|^2$$

$$\leqq \int_{P e_{[c]}} \Big\{ \sum_{\rho=1}^{\nu+\varkappa} \sqrt{\varphi_\rho(\xi)} \Big\}^2 \| d_\xi e \, c \|^2 = \| P \sum_{\rho=1}^{\nu+\varkappa} \mathfrak{b}_\rho \|^2$$

for every $P \in \mathfrak{R}$ and $\varkappa = 1, 2, \ldots$. Thus, making $\varkappa$ tend to $+\infty$, we obtain $\| P a_\nu \| \leqq \| P \mathfrak{b} \|$ for every $P \in \mathfrak{R}$, that is, $a_\nu \prec \mathfrak{b}$ $(\nu = 1, 2, \ldots)$ by definition.

For a sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$, if we can find two elements a, $\mathfrak{b} \in R$ such that for any $\varepsilon > 0$ there is ν_0 for which

$$a_\nu - a \prec \varepsilon \mathfrak{b} \qquad \text{for } \nu \geqq \nu_0,$$

then we shall say that $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ is <u>ring convergent</u> by $\mathfrak{R}$ to a limit a, denoting by

$$(\mathfrak{R})\text{-}\lim_{\nu\to\infty} a_\nu = a.$$

Such a limit a is determined uniquely, if exists, because it is evident by definition that $(\mathfrak{R})\text{-}\lim_{\nu\to\infty} a_\nu = a$ implies $\lim_{\nu\to\infty} a_\nu = a$.

Recalling the formulas (3) and (5), we have obviously

<u>Theorem</u> 21.4. $(\mathfrak{R})\text{-}\lim_{\nu\to\infty} a_\nu = a$ <u>implies</u>

$$(\mathfrak{R})\text{-}\lim_{\nu\to\infty} \alpha a_\nu = \alpha a \qquad \text{<u>for every complex number</u } \alpha,$$

$$(\mathfrak{R})\text{-}\lim_{\nu\to\infty} P a_\nu = P a \qquad \text{<u>for every</u } P \in \mathfrak{R}.$$

<u>Theorem</u> 21.5. $(\mathfrak{R})\text{-}\lim_{\nu\to\infty} a_\nu = a$, $(\mathfrak{R})\text{-}\lim_{\nu\to\infty} \mathfrak{b}_\nu = \mathfrak{b}$ <u>implies</u>

$$(\mathfrak{R})\text{-}\lim_{\nu\to\infty} (a_\nu + \mathfrak{b}_\nu) = a + \mathfrak{b}.$$

<u>Proof</u>. If $(\mathfrak{R})\text{-}\lim_{\nu\to\infty} a_\nu = a$, $(\mathfrak{R})\text{-}\lim_{\nu\to\infty} \mathfrak{b}_\nu = \mathfrak{b}$, then we can find by definition $\mathfrak{h}$, $\mathfrak{k} \in R$ such that for any $\varepsilon > 0$ there is ν_0 for which

$$a_\nu - a \prec \varepsilon \mathfrak{h}, \quad \mathfrak{b}_\nu - \mathfrak{b} \prec \varepsilon \mathfrak{k} \qquad \text{for } \nu \geqq \nu_0.$$

For such $\mathfrak{h}$, $\mathfrak{k}$, there exists by Theorem 21.3 $\mathfrak{l} \in R$ such that $\mathfrak{h} \prec \mathfrak{l}$ and $\mathfrak{k} \prec \mathfrak{l}$. Then we have by (1) and (3)

$$a_\nu - a \prec \varepsilon \mathfrak{l}, \quad \mathfrak{b}_\nu - \mathfrak{b} \prec \varepsilon \mathfrak{l} \qquad \text{for } \nu \geqq \nu_0,$$

and hence by (7)

$$(a_\nu + \mathfrak{b}_\nu) - (a + \mathfrak{b}) \prec 2\varepsilon \mathfrak{l} \qquad \text{for } \nu \geqq \nu_0.$$

Therefore we obtain $(\mathfrak{R})\text{-}\lim_{\nu\to\infty} (a_\nu + \mathfrak{b}_\nu) = a + \mathfrak{b}$ by definition.

<u>Theorem</u> 21.6. <u>In order that</u> $\lim_{\nu\to\infty} a_\nu = a$, <u>it is necessary and</u>

sufficient that every subsequence from a_ν $(\nu = 1, 2, \ldots)$ includes a subsequence which is ring convergent by $\mathfrak{R}$ to a.

Proof. If we have not $\lim_{\nu\to\infty} a_\nu = a$, then we can find a subsequence a_{ν_μ} $(\mu = 1, 2, \ldots)$ and $\varepsilon > 0$ such that

$$\| a_{\nu_\mu} - a \| \geqq \varepsilon \qquad \text{for every } \mu = 1, 2, \ldots .$$

Then a_{ν_μ} $(\mu = 1, 2, \ldots)$ include no subsequence which is ring convergent to a, because ring convergence implies convergence.

If $\lim_{\nu\to\infty} a_\nu = a$, then from any subsequence we can select a subsequence a_{ν_μ} $(\mu = 1, 2, \ldots)$ such that

$$\| a_{\nu_\mu} - a \| \leqq \frac{1}{\mu\, 2^\mu} \qquad (\mu = 1, 2, \ldots).$$

For such a_{ν_μ} $(\mu = 1, 2, \ldots)$, since

$$\sum_{\mu=1}^{\infty} \mu \| a_{\nu_\mu} - a \| \leqq \sum_{\mu=1}^{\infty} \frac{1}{2^\mu} = 1,$$

we can find by Theorem 21.3 $\ell \in R$ such that

$$\mu (a_{\nu_\mu} - a) \prec \ell \qquad \text{for every } \mu = 1, 2, \ldots .$$

and hence by (3)

$$a_{\nu_\mu} - a \prec \frac{1}{\mu}\ell \qquad (\mu = 1, 2, \ldots).$$

Therefore we have $(\mathfrak{R})\text{-}\lim_{\mu\to\infty} a_{\nu_\mu} = a$ by definition.

Theorem 21.7. For a sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$, if we can find a sequence $\ell_1 \succ \ell_2 \succ \cdots$ such that

$$\lim_{\nu\to\infty} \ell_\nu = 0, \quad a_\nu - a \prec \ell_\nu \qquad (\nu = 1, 2, \ldots),$$

then we have $(\mathfrak{R})\text{-}\lim_{\nu\to\infty} a_\nu = a$.

Proof. Since $\lim_{\nu\to\infty} \ell_\nu = 0$ by assumption, there exists by Theorem 21.6 $\ell \in R$ such that for any $\varepsilon > 0$ we can find ν_ε for which we have $\ell_{\nu_\varepsilon} \prec \varepsilon\, \ell$. Then we have for $\nu \geqq \nu_\varepsilon$

$$a_\nu - a \prec \ell_\nu \prec \ell_{\nu_\varepsilon} \prec \varepsilon\, \ell.$$

Therefore we obtain $(\mathfrak{R})\text{-}\lim_{\nu\to\infty} a_\nu = a$ by definition.

Theorem 21.8. In order that we have

$$\lim_{\nu\to\infty} \left(\frac{\ell_\nu}{a}, \mathfrak{P}\right) = \left(\frac{\ell}{a}, \mathfrak{P}\right)$$

in an open set being dense in $\bigcup C_{[a]}$, it is necessary and sufficient

that we can find a sequence $\mathcal{R} \ni P_\mu \uparrow_{\mu=1}^{\infty} C_{[a]}$ such that

$$(\mathcal{R})\text{-}\lim_{\nu\to\infty} P_\mu[a]f_\nu = P_\mu[a]f \quad \text{for every } \mu = 1, 2, \ldots .$$

Proof. We suppose firstly

$$\lim_{\nu\to\infty} \left(\frac{f_\nu}{a}, \mathfrak{x}\right) = \left(\frac{f}{a}, \mathfrak{x}\right)$$

in an open set A being dense in $\mathfrak{U}_{C_{[a]}}$. Since

$$\mathfrak{U}_{C_{[a]}} = A^- = \left\{\sum_{\mathfrak{U}_P \subset A} \mathfrak{U}_P\right\}^-,$$

we obtain by Theorem 15.7 $C_{[a]} = \bigcup_{\mathfrak{U}_P \subset A} P$, and hence we can find by Theorem 14.3 a sequence $P_\nu \in \mathcal{R}$ $(\nu = 1, 2, \ldots)$ such that $\mathfrak{U}_{P_\nu} \subset A$ for every $\nu = 1, 2, \ldots$, and

$$[a] = C_{[a]}[a] = \bigcup_{\nu=1}^{\infty} P_\nu[a].$$

This relation yields by §14(16')

$$C_{[a]} = \bigcup_{\nu=1}^{\infty} P_\nu C_{[a]} = \bigcup_{\nu=1}^{\infty} P_\nu .$$

Therefore we can find a sequence $\mathcal{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} C_{[a]}$ such that

$$\mathfrak{U}_{P_\nu} \subset A \qquad \text{for every } \nu = 1, 2, \ldots .$$

For an arbitrary $\varepsilon > 0$, putting

$$A_\nu = \prod_{\rho \geq \nu} \left\{\mathfrak{x} : \left|\left(\frac{f_\rho}{a}, \mathfrak{x}\right) - \left(\frac{f}{a}, \mathfrak{x}\right)\right| < \varepsilon \right\} \mathfrak{U}_{P_\nu} \qquad (\nu = 1, 2, \ldots),$$

we obtain a sequence of open sets $A_1 \subset A_2 \subset \ldots$, and we have obviously

$$\sum_{\nu=1}^{\infty} \mathfrak{U}_{P_\nu} \subset \sum_{\nu=1}^{\infty} A_\nu .$$

By virtue of Theorems 15.9 and 15.6, we can find further a sequence of projection operators $Q_\nu \in \mathcal{R}$ $(\nu = 1, 2, \ldots)$ such that $\mathfrak{U}_{Q_\nu} = A_\nu^-$ for every $\nu = 1, 2, \ldots$, and we have

$$\sum_{\nu=1}^{\infty} \mathfrak{U}_{P_\nu} \subset \sum_{\nu=1}^{\infty} \mathfrak{U}_{Q_\nu} \subset \mathfrak{U}_{C_{[a]}} .$$

This relation yields obviously

$$\left(\sum_{\nu=1}^{\infty} \mathfrak{U}_{P_\nu}\right)^- \subset \left(\sum_{\nu=1}^{\infty} \mathfrak{U}_{Q_\nu}\right)^- \subset \mathfrak{U}_{C_{[a]}} .$$

Since $P_\nu \uparrow_{\nu=1}^{\infty} C_{[a]}$, we conclude hence by Theorem 15.7

$$Q_\nu \uparrow_{\nu=1}^{\infty} C_{[a]}$$

and it is obvious that

$$\left|\left(\frac{f_\rho}{a}, \mathfrak{x}\right) - \left(\frac{f}{a}, \mathfrak{x}\right)\right| \leq \varepsilon \qquad \text{for } \mathfrak{x} \in \mathfrak{U}_{Q_\nu},\ \rho \geq \nu .$$

Since $\varepsilon > 0$ may be arbitrary, we can find therefore

$$\mathcal{R} \ni Q_{\mu,\nu} \uparrow_{\nu=1}^{\infty} C_{[a]} \qquad (\mu = 1, 2, \ldots)$$

such that

$$|(\frac{\ell_\rho}{a}, \mathfrak{x}) - (\frac{\ell}{a}, \mathfrak{x})| \leqq \frac{1}{\mu} \qquad \text{for } \mathfrak{x} \in \mathfrak{U}_{Q_{\mu,\nu}},\ \rho \geqq \nu.$$

Recalling Theorem 14.4, we can find then $\nu_{\mu,\sigma}$ $(\mu, \sigma = 1, 2, \ldots)$ such that

$$\bigcap_{\mu=1}^{\infty} Q_{\mu,\nu_{\mu,\sigma}}[a] \uparrow_{\sigma=1}^{\infty} C_{[a]}[a] = [a],$$

and hence by §14(16')

$$\bigcap_{\mu=1}^{\infty} Q_{\mu,\nu_{\mu,\sigma}} \uparrow_{\sigma=1}^{\infty} C_{[a]}.$$

Thus, putting $P_\sigma = \bigcap_{\mu=1}^{\infty} Q_{\mu,\nu_{\mu,\sigma}}$ $(\sigma = 1, 2, \ldots)$, we have

$$P_\sigma \uparrow_{\sigma=1}^{\infty} C_{[a]},$$

$$|(\frac{\ell_\rho}{a}, \mathfrak{x}) - (\frac{\ell}{a}, \mathfrak{x})| \leqq \frac{1}{\mu} \qquad \text{for } \mathfrak{x} \in \mathfrak{U}_{P_\sigma},\ \rho \geqq \nu_{\mu,\sigma}.$$

This relation yields by Theorems 17.3, 17.6 and the formula §19(10)

$$\|P P_\sigma[a]\ell_\rho - P P_\sigma[a]\ell\|^2 = \int_{P P_\sigma} |(\frac{\ell_\rho}{a}, \mathfrak{x}) - (\frac{\ell}{a}, \mathfrak{x})|^2 \|d\mathfrak{x}\, a\|^2 \leqq \frac{1}{\mu^2}\|P P_\sigma a\|^2$$

for every $P \in \mathcal{R}$ and $\rho \geqq \nu_{\mu,\sigma}$. Therefore we have by definition

$$(\mathcal{R})\text{-}\lim_{\rho\to\infty} P_\sigma[a]\ell_\rho = P_\sigma[a]\ell \qquad (\sigma = 1, 2, \ldots).$$

Conversely, if $(\mathcal{R})\text{-}\lim_{\nu\to\infty} P_\mu[a]\ell_\nu = P_\mu[a]\ell$ $(\mu = 1, 2, \ldots)$ for some sequence

$$\mathcal{R} \ni P_\mu \uparrow_{\mu=1}^{\infty} C_{[a]},$$

then we can find by definition $\ell_\mu \in R$ $(\mu = 1, 2, \ldots)$ such that for any $\varepsilon > 0$ there exists ν_μ for which

$$P_\mu[a]\ell_\nu - P_\mu[a]\ell \prec \varepsilon\, \ell_\mu \qquad \text{for } \nu \geqq \nu_\mu.$$

This relation yields by the formulas (6), (7) and (12) in §19 for every $\mathfrak{x} \in \mathfrak{U}_{P_\mu}$ and $\nu \geqq \nu_\mu$

$$|(\frac{\ell_\nu - \ell}{a}, \mathfrak{x})| = \lim_{P\to\mathfrak{x}} \frac{\|P(P_\mu[a]\ell_\nu - P_\mu[a]\ell)\|}{\|P a\|} \leqq \lim_{P\to\mathfrak{x}} \frac{\|\varepsilon P \ell_\mu\|}{\|P a\|} = \varepsilon |(\frac{\ell_\mu}{a}, \mathfrak{x})|$$

Recalling Theorem 15.10, we see easily that there exists an open set A_0 being dense in $\mathfrak{U}_{C_{[a]}}$ such that all $(\frac{\ell_\nu}{a}, \mathfrak{x})$, $(\frac{\ell}{a}, \mathfrak{x})$, $(\frac{\ell_\mu}{a}, \mathfrak{x})$ $(\nu, \mu = 1, 2, \ldots)$ are finite in A_0. Then we have by §19(5)

$$|(\frac{\ell_\nu}{a}, \mathfrak{x}) - (\frac{\ell}{a}, \mathfrak{x})| \leqq \varepsilon |(\frac{\ell_\mu}{a}, \mathfrak{x})|$$

for every $\mathfrak{x} \in \mathfrak{U}_{P_\mu} A_0$, $\nu \geq \nu_\mu$, and hence

$$\lim_{\nu \to \infty} \left(\frac{f_\nu}{a}, \mathfrak{x}\right) = \left(\frac{f}{a}, \mathfrak{x}\right) \qquad \text{for } \mathfrak{x} \in \sum_{\mu=1}^{\infty} \mathfrak{U}_{P_\mu} A_0.$$

Here $\sum_{\mu=1}^{\infty} \mathfrak{U}_{P_\mu} A_0$ is dense in $\mathfrak{U}_{\mathfrak{C}_{[a]}}$. In fact we have by Theorem 15.7

$$\left(\sum_{\mu=1}^{\infty} \mathfrak{U}_{P_\mu} A_0\right)^- = \left(\left(\sum_{\mu=1}^{\infty} \mathfrak{U}_{P_\mu}\right) A_0^-\right)^- = \left(\sum_{\mu=1}^{\infty} \mathfrak{U}_{P_\mu}\right)^- = \mathfrak{U}_{\mathfrak{C}_{[a]}},$$

because $A_0^- = \mathfrak{U}_{\mathfrak{C}_{[a]}}$.

Theorem 21.9. For a bounded continuous function φ on $\mathfrak{U}_P$, we have

$$(\mathfrak{K})\text{-}\lim_{\varepsilon \to 0} \sum_{\nu=1}^{\varkappa} \varphi(\mathfrak{x}_\nu) P_\nu a = \int_P \varphi(\mathfrak{x})\, d\mathfrak{x}\, a$$

for $P = \sum_{\nu=1}^{\varkappa} P_\nu$, $\sup_{\mathfrak{x}, \mathfrak{y} \in \mathfrak{U}_{P_\nu}} |\varphi(\mathfrak{x}) - \varphi(\mathfrak{y})| \leqq \varepsilon$,

$\mathfrak{x}_\nu \in \mathfrak{U}_{P_\nu}$ $(\nu = 1, 2, \ldots, \varkappa)$.

Proof. By virtue of Theorem 17.1 and the formula §17(4), we see easily that

$$\sum_{\nu=1}^{\varkappa} \varphi(\mathfrak{x}_\nu) P_\nu a - \int_P \varphi(\mathfrak{x})\, d\mathfrak{x}\, a \prec 2\varepsilon P a.$$

Thus we obtain our assertion by definition.

CHAPTER III

DILATAOR ANALYSIS

§22 Dilatators

Let $\mathfrak{R}$ be a closed ring of projection operators containing 1 in a Hilbert space R, and $\overline{\mathfrak{R}}$ the commutor of $\mathfrak{R}$. A linear operator T with domain D is said to be a dilatator by $\mathfrak{R}$ in R, if

1) D is dense in R : for any $x \in R$ we can find a sequence $x_\nu \in D$ $(\nu = 1, 2, \ldots)$ such that $\lim_{\nu\to\infty} x_\nu = x$,

2) T is commutative to $\overline{\mathfrak{R}}$: $Q \in \overline{\mathfrak{R}}$, $a \in D$ implies $Qa \in D$ and $TQa = QTa$,

3) T is closed : $\lim_{\nu\to\infty} a_\nu = a$, $\lim_{\nu\to\infty} Ta_\nu = l$ implies $a \in D$ and $Ta = l$.

Let T be a dilatator with domain D in the sequel.

Theorem 22.1. For an element $a \in D$, let a function $\varphi(\mathfrak{x})$ be integrable by a in a neighbourhood $\mathfrak{U}_P$ of the proper space $\mathfrak{E}_\varkappa$. In order that

$$\int_P \varphi(\mathfrak{x})\, d\mathfrak{x}\, a \in D,$$

it is necessary and sufficient that φ be integrable by Ta, and then

$$T\int_P \varphi(\mathfrak{x})\, d\mathfrak{x}\, a = \int_P \varphi(\mathfrak{x})\, d\mathfrak{x}\, Ta.$$

Proof. If φ is bounded continuous in $\mathfrak{U}_P$, then we have by definition

$$\lim_{\varepsilon\to 0} \sum_{\nu=1}^{\varkappa} \varphi(\mathfrak{x}_\nu) P_\nu a = \int_P \varphi(\mathfrak{x})\, d\mathfrak{x}\, a,$$

$$\lim_{\varepsilon\to 0} \sum_{\nu=1}^{\varkappa} \varphi(\mathfrak{x}_\nu) P_\nu Ta = \int_P \varphi(\mathfrak{x})\, d\mathfrak{x}\, Ta$$

for $P = \sum_{\nu=1}^{\varkappa} P_\nu$, $P_\nu \in \mathfrak{R}$, $\mathfrak{x}_\nu \in \mathfrak{U}_{P_\nu}$,

$$\sup_{\mathfrak{x},\mathfrak{y} \in \mathfrak{U}_{P_\nu}} |\varphi(\mathfrak{x}) - \varphi(\mathfrak{y})| \leq \varepsilon \qquad (\nu = 1, 2, \ldots, \varkappa).$$

Since we have by the postulate 2)

$$T\sum_{\nu=1}^{\infty}\varphi(\mathfrak{x}_\nu)P_\nu a = \sum_{\nu=1}^{\infty}\varphi(\mathfrak{x}_\nu)P_\nu T a,$$

we obtain hence by the postulate 3)

$$\int_P \varphi(\mathfrak{x})d\mathfrak{x}\, a \in D,$$

$$T\int_P \varphi(\mathfrak{x})d\mathfrak{x}\, a = \int_P \varphi(\mathfrak{x})d\mathfrak{x}\, Ta.$$

In general, if φ is integrable by Ta in $\mathcal{U}_P$, then we can find by definition a sequence $\mathcal{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P$ such that φ is bounded continuous in $\mathcal{U}_{P_\nu}$ for every $\nu = 1, 2, \ldots$, and

$$\lim_{\nu\to\infty}\int_{P_\nu}\varphi(\mathfrak{x})d\mathfrak{x}\, a = \int_P \varphi(\mathfrak{x})d\mathfrak{x}\, a$$

$$\lim_{\nu\to\infty}\int_{P_\nu}\varphi(\mathfrak{x})d\mathfrak{x}\, Ta = \int_P \varphi(\mathfrak{x})d\mathfrak{x}\, Ta.$$

On the other hand, we have

$$\int_{P_\nu}\varphi(\mathfrak{x})d\mathfrak{x}\, a \in D$$

$$T\int_{P_\nu}\varphi(\mathfrak{x})d\mathfrak{x}\, a = \int_{P_\nu}\varphi(\mathfrak{x})d\mathfrak{x}\, Ta \qquad (\nu = 1, 2, \ldots),$$

as proved just above. Thus we also obtain by the postulate 3)

$$\int_P \varphi(\mathfrak{x})d\mathfrak{x}\, a \in D,$$

$$T\int_P \varphi(\mathfrak{x})d\mathfrak{x}\, a = \int_P \varphi(\mathfrak{x})d\mathfrak{x}\, Ta.$$

Conversely if $\int_P \varphi(\mathfrak{x})d\mathfrak{x}\, a \in D$, then for every $\mathcal{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} P$ such that φ is bounded continuous in $\mathcal{U}_{P_\nu}$ for every $\nu = 1, 2, \ldots$, we have by Theorem 17.4

$$\int_{P_\nu}\varphi(\mathfrak{x})d\mathfrak{x}\, Ta = T\int_{P_\nu}\varphi(\mathfrak{x})d\mathfrak{x}\, a = P_\nu T\int_P \varphi(\mathfrak{x})d\mathfrak{x}\, a,$$

and hence

$$\lim_{\nu\to\infty}\int_{P_\nu}\varphi(\mathfrak{x})d\mathfrak{x}\, Ta = PT\int_P \varphi(\mathfrak{x})d\mathfrak{x}\, a = T\int_P \varphi(\mathfrak{x})d\mathfrak{x}\, a.$$

Therefore φ is integrable by Ta in $\mathcal{U}_P$.

Theorem 22.2. For any $x \in R$, we can find $\mathcal{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} C_{[x]}$ such that $P_\nu x \in D$ for every $\nu = 1, 2, \ldots$.

Proof. For any $0 \neq x \in R$ we can find $a \in D$ such that

$$[x]R \ni a \neq 0,$$

because D is dense in R and $[x]a = 0$ implies by the formula §14 (7) $(a,x) = 0$, and hence $\|x-a\|^2 \geqq \|x\|^2$. Since the relative spectrum $(\frac{x}{a},\mathfrak{P})$ is by Theorem 18.4 integrable by a in $\mathfrak{U}_{\mathfrak{C}_{[a]}}$, we can find $P \in \mathfrak{R}$ such that $0 \neq P \leqq \mathfrak{C}_{[[a]x]}$ and $(\frac{x}{a},\mathfrak{P})$ is bounded in $\mathfrak{U}_P$. For such $P \in \mathfrak{R}$ we have by Theorem 22.1

$$P[a]x = \int_P (\tfrac{x}{a},\mathfrak{P})\, d\mathfrak{P}\, a \in D$$

and $P[a]x \neq 0$, because we have by the formula §14(16')

$$\mathfrak{C}_{[P[a]x]} = P\,\mathfrak{C}_{[[a]x]} = P \neq 0.$$

Since we have by the formula §14(1)

$$[a] = [[x]a] \leqq [x],$$

we obtain by Theorem 19.4 and the formula §14(3)

$$P[a]x = P\,\mathfrak{C}_{[a]}[x]x = P\,\mathfrak{C}_{[a]}x.$$

Thus we conclude easily by the formula §14(6') that we have

$$[x] = \bigcup_{Px \in D,\, P \in \mathfrak{R}} P[x] \qquad \text{for every } x \in R.$$

Since $Px,\ Qx \in D$; $P,\ Q \in \mathfrak{R}$ implies by Theorem 12.6

$$(P \vee Q)x = Px + Qx - PQx \in D,$$

we can find therefore by Theorem 14.3 a sequence $P_\nu \in \mathfrak{R}$ $(\nu = 1, 2, \ldots)$ such that $P_\nu x \in D$ $(\nu = 1, 2, \ldots)$ and $P_\nu[x] \uparrow_{\nu=1}^{\infty} [x]$. Then we have by the formulas (12), (15), (16'), and (17) in §14

$$P_\nu\,\mathfrak{C}_{[x]}x = P_\nu x \in D, \qquad P_\nu\,\mathfrak{C}_{[x]} = \mathfrak{C}_{P_\nu[x]} \uparrow_{\nu=1}^{\infty} \mathfrak{C}_{[x]}.$$

Theorem 22.3. *If a function $\varphi(\mathfrak{P})$ is finite continuous in an open dense set A of the proper space $\mathfrak{L}_{\mathfrak{R}}$, then, denoting by D the totality of those elements $x \in R$ by which φ is integrable in $\mathfrak{U}_{\mathfrak{C}_{[x]}}$, and putting*

$$Tx = \int_{\mathfrak{C}_{[x]}} \varphi(\mathfrak{P})\, d\mathfrak{P}\, x \qquad (x \in D),$$

we obtain a dilatator with domain D. Such a dilatator T will be denoted by

$$\int \varphi(\mathfrak{P})\, d\mathfrak{P}.$$

Proof. For every $a \in R$, φ is almost continuous in $\mathfrak{U}_{\mathfrak{C}_{[a]}}$.

In fact, we have obviously

$$A\mathfrak{U}_{\mathfrak{C}[a]} = \sum_{\mathfrak{U}_P \subset A\mathfrak{U}_{\mathfrak{C}[a]}} \mathfrak{U}_P,$$
$$(A\mathfrak{U}_{\mathfrak{C}[a]})^- = A^-\mathfrak{U}_{\mathfrak{C}[a]} = \mathfrak{U}_{\mathfrak{C}[a]}.$$

Thus we obtain by Theorem 15.7

$$\mathfrak{C}_{[a]} = \bigcup_{\mathfrak{U}_P \subset A\mathfrak{U}_{\mathfrak{C}[a]}} P,$$

and hence we can find by Theorem 14.7 a sequence $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} \mathfrak{C}_{[a]}$ such that $\mathfrak{U}_{P_\nu} \subset A$ for every $\nu = 1, 2, \ldots$. Then, since every $\mathfrak{U}_{P_\nu}$ is compact and φ is finite continuous in $\mathfrak{U}_{P_\nu}$, φ is bounded continuous in $\mathfrak{U}_{P_\nu}$ for every $\nu = 1, 2, \ldots$. Therefore φ is almost continuous in $\mathfrak{U}_{\mathfrak{C}[a]}$ for every $a \in R$. Furthermore, since φ is bounded continuous in $\mathfrak{U}_{P_\nu}$ and

$$\mathfrak{U}_{\mathfrak{C}[P_\nu a]} = \mathfrak{U}_{P_\nu}\mathfrak{U}_{\mathfrak{C}[a]}$$

by the formula §14(16'), we have obviously $P_\nu a \in D$ for every $\nu = 1, 2, \ldots$ and $\lim_{\nu\to\infty} P_\nu a = \mathfrak{C}_{[a]}a = a$. Therefore D is dense in R.

For $a, \mathfrak{k} \in R$, $\mathfrak{C}_{[a]} \leqq \mathfrak{C}_{[\mathfrak{k}]}$, we see easily by definition that φ is integrable by a in $\mathfrak{U}_{\mathfrak{C}[a]}$ if and only if φ is integrable by a in $\mathfrak{U}_{\mathfrak{C}[\mathfrak{k}]}$, and then by Theorem 17.4

$$\int_{\mathfrak{C}[a]} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} a = \int_{\mathfrak{C}[\mathfrak{k}]} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} a.$$

Thus we see easily by Theorem 17.7 that D is a linear manifold and T is a linear operator. Furthermore it is obvious by Theorem 17.8 that T is closed. Recalling Theorem 17.11, we see at once that $a \in D$ implies $Qa \in D$ for every $Q \in \overline{\mathfrak{R}}$ and

$$Q\int_{\mathfrak{C}[a]} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} a = \int_{\mathfrak{C}[a]} \varphi(\mathfrak{x})\, d_{\mathfrak{x}} Qa,$$

that is, $QTa = TQa$. Therefore T is a dilatator by defition.

For an operator T with domain D, if D is dense in R, then there exists uniquely an operator T_1 with domain D_1 such that

$$(Tx, y) = (x, T_1 y) \qquad \text{for } x \in D,\ y \in D_1,$$

and $(Tx, y) = (x, z)$ for every $x \in D$ implies $y \in D_1$ and $Ty = z$. Because $(x, z) = (x, z_1)$ for every $x \in D$ implies by Theorem 9.1 $z = z_1$. Such an operator T_1 is called the adjoint operator of T and denoted by T^*.

With this definition we have

Theorem 22.4. For a dilatator by $\mathcal{R}$:

$$T = \int \varphi(\xi)\, d\xi,$$

its adjoint operator T^* also is a dilatator by $\mathcal{R}$ with the same domain and

$$T^* = \int \overline{\varphi(\xi)}\, d\xi .$$

Proof. Putting $T_1 = \int \overline{\varphi(\xi)}\, d\xi$, we obtain by Theorem 22.3 a dilatator T_1 by $\mathcal{R}$, and recalling Theorem 17.3, we see that the domain of T_1 coincides with that of T. Furthermore, for every x, $y \in D$, putting $P = C_{[x]} \vee C_{[y]}$, we have by Theorem 17.9

$$(Tx, y) = (Tx, Py) = \int_P \varphi(\xi)\, (d_\xi x, y) = (x, T_1 y).$$

If $(Tx, a) = (x, \ell)$ for every $x \in D$, then we can find by Theorem 22.2 a sequence $\mathcal{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} C_{[a]}$ such that $P_\nu a \in D$ for every $\nu = 1, 2, \ldots$, and we have for every $x \in D$

$$(x, P_\nu \ell) = (P_\nu x, \ell) = (T P_\nu x, a) = (Tx, P_\nu a) = (x, T_1 P_\nu a).$$

This relation yields $P_\nu \ell = T_1 P_\nu a$ for every $\nu = 1, 2, \ldots$. Since

$$\lim_{\nu \to \infty} P_\nu a = C_{[a]} a = a, \qquad \lim_{\nu \to \infty} P_\nu \ell = C_{[a]} \ell,$$

we obtain hence $a \in D$ and $T_1 a = C_{[a]} \ell$, because T_1 is closed. On the other hand, we have for every $x \in D$

$$\begin{aligned}(x, C_{[a]} \ell) &= (C_{[a]} x, \ell) = (T C_{[a]} x, a) \\ &= (Tx, a) = (x, \ell),\end{aligned}$$

and hence $C_{[a]} \ell = \ell$. Therefore we obtain $T_1 = T^*$ by definition.

Theorem 22.5. For two dilatators by $\mathcal{R}$:

$$T_1 = \int \varphi_1(\xi)\, d\xi, \qquad T_2 = \int \varphi_2(\xi)\, d\xi,$$

if both $T_1 T_2 a$ and $T_1 a$ have any sense, then $T_2 T_1 a$ also has a sense and

$$T_1 T_2 a = T_2 T_1 a = \int_{C_{[a]}} \varphi_1(\xi)\, \varphi_2(\xi)\, d\xi a.$$

Proof. As proved in Proof of Theorem 22.3, we can find a sequence $\mathcal{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} C_{[a]}$ such that both φ_1 and φ_2 are bounded continuous in $\mathfrak{O}_{P_\nu}$ for every $\nu = 1, 2, \ldots$. For the adjoint operator

T_1^* of T_1 we have by Theorem 22.4 and the formulas (6), (7) in §17

$$(T_1 T_2 P_\nu a, x) = (T_2 P_\nu a, T_1^* P_\nu x)$$
$$= \int_{P_\nu} \varphi_1(\mathfrak{x})\varphi_2(\mathfrak{x})\,(d_{\mathfrak{x}} a, x) = \Big(\int_{P_\nu} \varphi_1(\mathfrak{x})\varphi_2(\mathfrak{x}) d_{\mathfrak{x}} a, x\Big)$$

for every $x \in R$, and hence

$$T_1 T_2 P_\nu a = \int_{P_\nu} \varphi_1(\mathfrak{x})\varphi_2(\mathfrak{x})\, d_{\mathfrak{x}} a.$$

We also can prove likewise

$$T_2 T_1 P_\nu a = \int_{P_\nu} \varphi_1(\mathfrak{x})\varphi_2(\mathfrak{x})\, d_{\mathfrak{x}} a.$$

If $T_1 T_2 a$ has any sense, then we have

$$T_1 T_2 P_\nu a = P_\nu T_1 T_2 a,$$

and hence by the formula §17(9)

$$\int_{P_\nu} |\varphi_1(\mathfrak{x})\varphi_2(\mathfrak{x})|^2 \|d_{\mathfrak{x}} a\|^2 = \|P_\nu T_1 T_2 a\|^2 \leqq \|T_1 T_2 a\|^2.$$

Thus we see by Theorem 17.3 that $\varphi_1(\mathfrak{x})\varphi_2(\mathfrak{x})$ is integrable by a in $\mathcal{O}_{\mathcal{C}_{[a]}}$ and we obtain by the formula §14(17)

$$\int_{\mathcal{C}_{[a]}} \varphi_1(\mathfrak{x})\varphi_2(\mathfrak{x})\, d_{\mathfrak{x}} a = \mathcal{C}_{[a]} T_1 T_2 a = T_1 T_2 a.$$

Furthermore, if $T_1 a$ has a sense, then we have

$$T_2 P_\nu T_1 a = T_2 T_1 P_\nu a = T_1 T_2 P_\nu a = P_\nu T_1 T_2 a,$$

and hence

$$\lim_{\nu\to\infty} P_\nu T_1 a = \mathcal{C}_{[a]} T_1 a = T_1 a,$$
$$\lim_{\nu\to\infty} T_2 P_\nu T_1 a = \mathcal{C}_{[a]} T_1 T_2 a = T_1 T_2 a.$$

Therefore, $T_2 T_1 a$ also has a sense, and $T_2 T_1 a = T_1 T_2 a$, because T_2 is closed.

Concerning continuous functions on the proper space $\mathfrak{C}_{\mathfrak{X}}$ we have

Theorem 22.6. *For a finite continuous function φ on a dense open set A of the proper space $\mathfrak{C}_{\mathfrak{X}}$, there exists uniquely a continuous function ψ on the whole space $\mathfrak{C}_{\mathfrak{X}}$ such that*

$$\psi(\mathfrak{x}) = \varphi(\mathfrak{x}) \qquad \text{for } \mathfrak{x} \in A.$$

Such ψ is called the *continuous extension* of φ over $\mathfrak{C}_{\mathfrak{X}}$.

Proof. We shall consider firstly the case where φ is a real

valued function. For each real number ξ, putting

$$A_\xi = \{\mathfrak{x}: \varphi(\mathfrak{x}) > \xi\},$$

we obtain an open set $A_\xi \subset A$ $(-\infty < \xi < +\infty)$, and hence its closure A_ξ^- also is open by Theorem 15.9. If we put

$$\psi(\mathfrak{x}) = \begin{cases} \sup_{\mathfrak{x} \in A_\xi^-} \xi & \text{for } \mathfrak{x} \in \sum_{-\infty<\xi<+\infty} A_\xi^- \\ -\infty & \text{for } \mathfrak{x} \bar{\in} \sum_{-\infty<\xi<+\infty} A_\xi^-, \end{cases}$$

then we have obviously

$$\{\mathfrak{x}: \psi(\mathfrak{x}) > \xi\} \subset A_\xi^- \subset \{\mathfrak{x}: \psi(\mathfrak{x}) \geqq \xi\},$$

because $A_\xi^- \supset A_\eta^-$ for $\xi < \eta$. From this relation we conclude

$$\{\mathfrak{x}: \psi(\mathfrak{x}) > \xi\} = \sum_{\eta>\xi} \{\mathfrak{x}: \psi(\mathfrak{x}) \geqq \eta\}$$
$$= \sum_{\eta>\xi} \{\mathfrak{x}: \psi(\mathfrak{x}) > \eta\} = \sum_{\eta>\xi} A_\eta^-,$$
$$\{\mathfrak{x}: \psi(\mathfrak{x}) \geqq \xi\} = \prod_{\eta<\xi} \{\mathfrak{x}: \psi(\mathfrak{x}) > \eta\}$$
$$= \prod_{\eta<\xi} \{\mathfrak{x}: \psi(\mathfrak{x}) \geqq \eta\} = \prod_{\eta<\xi} A_\eta^-.$$

Since A_η^- is open and closed for every real number η, we see hence that ψ is continuous over the whole space $\mathfrak{C}_\mathfrak{R}$. Furthermore, since φ is continuous in A by assumption, we have obviously

$$A A_\xi^- \subset \{\mathfrak{x}: \varphi(\mathfrak{x}) \geqq \xi\},$$

and hence

$$A\{\mathfrak{x}: \psi(\mathfrak{x}) > \xi\} \subset \{\mathfrak{x}: \varphi(\mathfrak{x}) \geqq \xi\},$$
$$\{\mathfrak{x}: \varphi(\mathfrak{x}) > \xi\} = A_\xi \subset A\{\mathfrak{x}: \psi(\mathfrak{x}) \geqq \xi\}.$$

From this relation we conclude $\varphi(\mathfrak{x}) = \psi(\mathfrak{x})$ for $\mathfrak{x} \in A$.

Theorem 22.7. *If a continuous function φ on the proper space $\mathfrak{C}_\mathfrak{R}$ is finite in a dense set, then φ is almost continuous in $\mathfrak{C}_\mathfrak{R}$, that is, we can find a sequence $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} 1$ such that φ is bounded in $\mathfrak{U}_{P_\nu}$ for every $\nu = 1, 2, \ldots$.*

Proof. Putting $A_\nu = \{\mathfrak{x}: |\varphi(\mathfrak{x})| < \nu\}$ $(\nu = 1, 2, \ldots)$, we obtain an open set A_ν, and hence its closure A_ν^- also is open by Theorem 15.9. Therefore we can find by Theorem 15.6 $P_\nu \in \mathfrak{R}$ for which

$$\mathfrak{U}_{P_\nu} = A_\nu^- \qquad (\nu = 1, 2, \ldots).$$

Then we have obviously $P_\nu \uparrow_{\nu=1}^{\infty} 1$ by Theorem 15.7, because φ is finite in a dense set. Furthermore we have

$$|\varphi(\mathfrak{x})| \leqq \nu \qquad \text{for } \mathfrak{x} \in \mathfrak{U}_{P_\nu} \quad (\nu = 1, 2, \ldots).$$

If a function φ is almost continuous in the whole space $\mathcal{C}_{\mathcal{R}}$, then we may denote by

$$\int \varphi(\mathfrak{x})\, d\mathfrak{x}\, \alpha$$

the integral of φ by α in $\mathcal{C}_{\mathcal{R}}$, that is,

$$\int \varphi(\mathfrak{x})\, d\mathfrak{x}\, \alpha = \int_1 \varphi(\mathfrak{x})\, d\mathfrak{x}\, \alpha.$$

We see then easily by definition that φ is integrable by α in $\mathcal{C}_{[a]}$ if and only if φ is integrable by α in $\mathcal{C}_{\mathcal{R}}$, and then we have

$$\int_{\mathcal{C}_{[a]}} \varphi(\mathfrak{x})\, d\mathfrak{x}\, \alpha = \int \varphi(\mathfrak{x})\, d\mathfrak{x}\, \alpha.$$

§23 Spectralities

Let φ be a continuous function on the proper space $\mathcal{C}_{\mathcal{R}}$ and finite in an open dense set. For an arbitrary point set $\mathfrak{E}$ in the complex plane G_τ, we can find by Theorems 15.9 and 15.6 $E^*(\mathfrak{E}) \in \mathcal{R}$ and $E_*(\mathfrak{E}) \in \mathcal{R}$ such that

(1) $$U_{E^*(\mathfrak{E})} = \{\mathfrak{x}: \ \varphi(\mathfrak{x}) \in \mathfrak{E}\}^{-\circ},$$

(2) $$U_{E_*(\mathfrak{E})} = \{\mathfrak{x}: \ \varphi(\mathfrak{x}) \in \mathfrak{E}\}^{\circ-},$$

and we have by the relation §15(4)

(3) $$E^*(\mathfrak{E}) \geqq E_*(\mathfrak{E}) \qquad \text{for every } \mathfrak{E}.$$

Furthermore we obtain immediately by definition

(4) $$E^*(\mathfrak{E}_1) \geqq E^*(\mathfrak{E}_2), \quad E_*(\mathfrak{E}_1) \geqq E_*(\mathfrak{E}_2) \quad \text{for } \mathfrak{E}_1 > \mathfrak{E}_2,$$

(5) $$E^*(0) = E_*(0) = 0,$$

(6) $$E^*(G_\tau) = E_*(G_\tau) = 1 \quad \text{for the whole plane } G_\tau,$$

because φ is finite in an open dense set.

We have obviously by definition

$$\{\mathfrak{x}: \ \varphi(\mathfrak{x}) \in \mathfrak{E}\}\{\mathfrak{x}: \ \varphi(\mathfrak{x}) \in \mathfrak{E}'\} = 0.$$

From this relation we conclude easily

$$U_{E^*(\mathfrak{E})} U_{E_*(\mathfrak{E}')} = 0,$$

and hence $E^*(\mathfrak{E})E_*(\mathfrak{E}') = 0$ by §15(5). Therefore we have by (4)

(7) $$E^*(\mathfrak{E})E_*(\mathfrak{F}) = 0 \qquad \text{for } \mathfrak{E}\mathfrak{F} = 0.$$

Since φ is finite in an open dense set, we have for every point set $\mathfrak{E}$ in $G_\mathfrak{r}$

$$(\{\mathfrak{z}:\ \varphi(\mathfrak{z})\in\mathfrak{E}\}\dot{+}\{\mathfrak{z}:\ \varphi(\mathfrak{z})\in\mathfrak{E}'\})^{-0}=\mathfrak{C}_\mathfrak{R}.$$

From this relation we conclude easily by the definition (1), (2)

$$U_{E^*(\mathfrak{E})}\dot{+}U_{E_*(\mathfrak{E}')}=\mathfrak{C}_\mathfrak{R}.$$

Therefore we obtain by the formulas (7) and §15(3)

$$E^*(\mathfrak{E})+E_*(\mathfrak{E}')=1. \tag{8}$$

For a sequence of point sets $\mathfrak{E}_\nu$ $(\nu=1,2,\dots)$ in $G_\mathfrak{r}$, putting

$$A_\nu=\{\mathfrak{z}:\ \varphi(\mathfrak{z})\in\mathfrak{E}_\nu\}\qquad(\nu=1,2,\dots),$$

we have

$$\{\mathfrak{z}:\ \varphi(\mathfrak{z})\in\sum_{\nu=1}^{\infty}\mathfrak{E}_\nu\}=\sum_{\nu=1}^{\infty}A_\nu\subset\sum_{\nu=1}^{\infty}A_\nu^-$$
$$=\sum_{\nu=1}^{\infty}A_\nu^{-0}\dot{+}\sum_{\nu=1}^{\infty}A_\nu^-(A_\nu^{-0})'.$$

Here $A_\nu^-(A_\nu^{-0})'$ is nowhere dense, because

$$(A_\nu^-(A_\nu^{-0})')^{-0}=A_\nu^{-0}(A_\nu^{-0})'^{-0}\subset A_\nu^{-0-}(A_\nu^{-0-})'=0.$$

Consequently $\sum_{\nu=1}^{\infty}A_\nu^-(A_\nu^{-0})'$ also is nowhere dense by Theorem 15.10.

Therefore we have

$$\{\mathfrak{z}:\ \varphi(\mathfrak{z})\in\sum_{\nu=1}^{\infty}\mathfrak{E}_\nu\}^{-0}$$
$$\subset(\sum_{\nu=1}^{\infty}A_\nu^{-0})^-\dot{+}(\sum_{\nu=1}^{\infty}A_\nu^-(A_\nu^{-0})')^{-0}=(\sum_{\nu=1}^{\infty}A_\nu^{-0})^-,$$

that is,

$$U_{E^*(\sum_{\nu=1}^{\infty}\mathfrak{E}_\nu)}\subset(\sum_{\nu=1}^{\infty}U_{E^*(\mathfrak{E}_\nu)})^-.$$

On the other hand we have obviously by (4)

$$U_{E^*(\sum_{\nu=1}^{\infty}\mathfrak{E}_\nu)}\supset U_{E^*(\mathfrak{E}_\mu)}\quad\text{for every }\mu=1,2,\dots,$$

and hence

$$U_{E^*(\sum_{\nu=1}^{\infty}\mathfrak{E}_\nu)}\supset(\sum_{\nu=1}^{\infty}U_{E^*(\mathfrak{E}_\nu)})^-.$$

Thus we obtain by Theorem 15.7

$$E^*(\sum_{\nu=1}^{\infty}\mathfrak{E}_\nu)=\bigcup_{\nu=1}^{\infty}E^*(\mathfrak{E}_\nu). \tag{9}$$

From this formula (9) we conclude immediately by (8)

$$E_*(\prod_{\nu=1}^{\infty}\mathfrak{E}_\nu)=\bigcap_{\nu=1}^{\infty}E_*(\mathfrak{E}_\nu). \tag{10}$$

A point set $\mathfrak{E}$ in $G_\mathfrak{r}$ is said to be <u>measurable</u> by φ, if

$$E^*(\mathfrak{E})=E_*(\mathfrak{E}),$$

that is, if $\{\mathfrak{z}:\ \varphi(\mathfrak{z})\in\mathfrak{E}\}^{-0}=\{\mathfrak{z}:\ \varphi(\mathfrak{z})\in\mathfrak{E}\}^{0-}$, and ther

this projection operator is called the spectrality of $\mathfrak{E}$ by φ and denoted by $E(\mathfrak{E})$ or by $E_{\varphi}(\mathfrak{E})$, when we need indicate φ.

Concerning measurable sets, we see at once by (8) that if $\mathfrak{E}$ is measurable, then its complement $\mathfrak{E}'$ also is measurable and

$$E(\mathfrak{E}) + E(\mathfrak{E}') = 1. \tag{11}$$

If every $\mathfrak{E}_\nu$ $(\nu = 1, 2, \ldots)$ is measurable, then we have by the formulas (9) and (4)

$$E^*(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu) = \bigcup_{\nu=1}^{\infty} E(\mathfrak{E}_\nu) \leqq E_*(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu),$$

that is, $\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu$ also is measurable and

$$E(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu) = \bigcup_{\nu=1}^{\infty} E(\mathfrak{E}_\nu). \tag{12}$$

From this formula (12) we conclude immediately by (11) that if every $\mathfrak{E}_\nu$ $(\nu = 1, 2, \ldots)$ is measurable, then $\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu$ also is measurable and

$$E(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu) = \bigcap_{\nu=1}^{\infty} E(\mathfrak{E}_\nu). \tag{13}$$

Thus we obtain by (4) for measurable sets

$$\mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \ldots \text{ implies } E(\mathfrak{E}_\nu) \uparrow_{\nu=1}^{\infty} E(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu), \tag{14}$$

$$\mathfrak{E}_1 \supset \mathfrak{E}_2 \supset \ldots \text{ implies } E(\mathfrak{E}_\nu) \downarrow_{\nu=1}^{\infty} E(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu). \tag{15}$$

Furthermore we have obviously by the definition (1), (2)

$$\{\mathfrak{x}: \varphi(\mathfrak{x}) \in \mathfrak{E}^0\}^- \subset \mathfrak{U}_{E(\mathfrak{E})} \subset \{\mathfrak{x}: \varphi(\mathfrak{x}) \in \mathfrak{E}^-\}^0. \tag{16}$$

Recalling Theorem 12.6, for two measurable sets $\mathfrak{E}$ and $\mathfrak{F}$ we obtain by the formula (12)

$$E(\mathfrak{E} \dotplus \mathfrak{F}) = E(\mathfrak{E}) + E(\mathfrak{F}) - E(\mathfrak{E})E(\mathfrak{F}),$$

and hence by (13)

$$E(\mathfrak{E} \dotplus \mathfrak{F}) + E(\mathfrak{E}\mathfrak{F}) = E(\mathfrak{E}) + E(\mathfrak{F}). \tag{17}$$

We shall denote by Λ^0 an arbitrary open set and by Λ^- an arbitrary closed set in the complex plane $\mathfrak{G}$. Since φ is continuous by assumption, for every open set Λ^0 in $\mathfrak{G}$

$$\{\mathfrak{x}: \varphi(\mathfrak{x}) \in \Lambda^0\}$$

is open, and hence by Theorem 15.9

$$\{\mathfrak{x}: \varphi(\mathfrak{x}) \in \Lambda^0\}^{-0} = \{\mathfrak{x}: \varphi(\mathfrak{x}) \in \Lambda^0\}^- = \{\mathfrak{x}: \varphi(\mathfrak{x}) \in \Lambda^{0-}\}$$

Therefore every open set in $\mathfrak{G}$ is measurable, and consequently every closed set in $\mathfrak{G}$ also is measurable.

The projection operator valued function $E(\mathfrak{E})$ for all measurable sets $\mathfrak{E}$ by φ may be often called the spectrality of φ. A point set $\mathfrak{E}$ is said to be regularly measurable by φ, if

$$\bigcup_{\mathfrak{E} \succ \Lambda^-} E(\Lambda^-) = \bigcap_{\mathfrak{E} \subset \Lambda^\circ} E(\Lambda^\circ)$$

for the spectrality E of φ. If $\mathfrak{E}$ is regularly measurable, then $\mathfrak{E}$ is measurable, because we have by (4)

$$E^*(\mathfrak{E}) \leqq \bigcap_{\mathfrak{E} \subset \Lambda^\circ} E(\Lambda^\circ) = \bigcup_{\mathfrak{E} \succ \Lambda^-} E(\Lambda^-) \leqq E_*(\mathfrak{E}),$$

and hence $E^*(\mathfrak{E}) = E_*(\mathfrak{E})$ by (3). Furthermore we have

$$(18) \qquad E(\mathfrak{E}) = \bigcup_{\mathfrak{E} \succ \Lambda^-} E(\Lambda^-) = \bigcap_{\mathfrak{E} \subset \Lambda^\circ} E(\Lambda^\circ)$$

if $\mathfrak{E}$ is regularly measurable.

For every point set $\mathfrak{E}$ in G we have by the formula (11)

$$\bigcup_{\mathfrak{E}' \succ \Lambda^-} E(\Lambda^-) = \bigcup_{\mathfrak{E}' \subset \Lambda^\circ} E(\Lambda^{\circ\prime}) = 1 - \bigcap_{\mathfrak{E} \subset \Lambda^\circ} E(\Lambda^\circ),$$

$$\bigcap_{\mathfrak{E}' \subset \Lambda^\circ} E(\Lambda^\circ) = \bigcap_{\mathfrak{E}' \succ \Lambda^-} E(\Lambda^{-\prime}) = 1 - \bigcup_{\mathfrak{E} \succ \Lambda^-} E(\Lambda^-).$$

Thus, if $\mathfrak{E}$ is regularly measurable, then its complement $\mathfrak{E}'$ also is regularly measurable.

For a sequence of regularly measurable sets $\mathfrak{E}_\nu$ $(\nu = 1, 2, \ldots)$, we have by the formula (3)

$$E\Big(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu\Big) = \bigcap_{\nu=1}^{\infty} E(\mathfrak{E}_\nu) = \bigcap_{\nu=1}^{\infty} \bigcap_{\mathfrak{E}_\nu \subset \Lambda^\circ} E(\Lambda^\circ) \geqq \bigcap_{\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu \subset \Lambda^\circ} E(\Lambda^\circ),$$

and hence by (4)

$$E\Big(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu\Big) = \bigcap_{\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu \subset \Lambda^\circ} E(\Lambda^\circ).$$

Since $E(\mathfrak{E}_\nu) = \bigcup_{\mathfrak{E}_\nu \succ \Lambda^-} E(\Lambda^-)$ by assumption, for any $\alpha \in R$ we can find by Theorem 14.3 a sequence of closed sets

$$\Lambda^-_{\nu,1} \subset \Lambda^-_{\nu,2} \subset \cdots, \qquad \Lambda^-_{\nu,\mu} \subset \mathfrak{E}_\nu \quad (\nu, \mu = 1, 2, \ldots)$$

such that

$$E(\Lambda^-_{\nu,\mu})[\alpha] \uparrow_{\mu=1}^{\infty} E(\mathfrak{E}_\nu)[\alpha] \qquad (\nu = 1, 2, \ldots),$$

and hence

$$E(\Lambda^-_{\nu,\mu}) E\Big(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu\Big)[\alpha] \uparrow_{\mu=1}^{\infty} E\Big(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu\Big)[\alpha] \quad (\nu = 1, 2, \ldots).$$

Then we can find further by Theorem 14.4 $\mu_{\nu,\rho}$ $(\nu, \rho = 1, 2, \ldots)$ such that

$$\bigcap_{\nu=1}^{\infty} E(\Lambda_{\nu,\mu^-_{\nu,\rho}}) E\Big(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu\Big)[\alpha] \uparrow_{\rho=1}^{\infty} E\Big(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu\Big)[\alpha],$$

and hence by (13)

$$E\Big(\prod_{\nu=1}^{\infty} \Lambda_{\nu,\mu^-_{\nu,\rho}}\Big)[\alpha] \uparrow_{\rho=1}^{\infty} E\Big(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu\Big)[\alpha],$$

because $E(\prod_{\nu=1}^{\infty} \Lambda_{\nu,\mu_{\nu,\rho}}^{-}) \leqq E(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu)$ by (4). Therefore we obtain

$$\bigcup_{\prod_{\nu=1}^{\infty}\mathfrak{E}_\nu > \Lambda^-} E(\Lambda^-)[\alpha] = E(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu)[\alpha].$$

Here $\alpha \in R$ may be arbitrary, we obtain hence by Theorem 14.1

$$E(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu) = \bigcup_{\prod_{\nu=1}^{\infty}\mathfrak{E}_\nu > \Lambda^-} E(\Lambda^-).$$

Thus $\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu$ is regularly measurable. Consequently $\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu$ also is regularly measurable, because $\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu = (\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu')'$.

For an open set Γ^0 in the complex plane G_ζ, we can find a sequence of closed sets Λ_ν^- $(\nu = 1, 2, \ldots)$ such that

$$\Gamma^0 = \sum_{\nu=1}^{\infty} \Lambda_\nu^-,$$

and hence $E(\Gamma^0) = \bigcup_{\nu=1}^{\infty} E(\Lambda_\nu^-)$ by (12). Thus we have obviously by (4)

$$E(\Gamma^0) = \bigcup_{\Gamma^0 > \Lambda^-} E(\Lambda^-)$$

Therefore every open set in G_ζ is regularly measurable by definition.

Now we can state hence

Theorem 23.1. The totality of regularly measurable sets by φ constitutes a totally additive set class including all Borel sets.

For two operators T_1 and T_2 with domain D_1 and D_2 respectively, we define $T_1 + T_2$ to mean an operator with domain $D_1 D_2$ and

$$(T_1 + T_2)x = T_1 x + T_2 x \qquad \text{for } x \in D_1 D_2.$$

We also define $T_1 T_2$ to mean that the domain of $T_1 T_2$ is composed of those elements x for which $T_2 x \in D_1$, $x \in D_2$, and

$$T_1 T_2 x = T_1(T_2 x).$$

For an operator T we define T^ν $(\nu = 1, 2, \ldots)$ to mean

$$T^1 = T, \quad T^{\nu+1} = T(T^\nu) \qquad \text{for } \nu = 1, 2, \ldots .$$

Theorem 23.2. Let E be the spectrality of φ. Putting

$$T = \int \varphi(\xi)\, d\xi,$$

$$\mathfrak{E}_{\alpha,\varepsilon} = \{\zeta : |\zeta - \alpha| \leqq \varepsilon\}$$

for a complex number α and a positive number ε, the closed linear manifold $E(\mathfrak{E}_{\alpha,\varepsilon})R$ is composed of those elements x for which

$$\sup_{\nu \geqq 1} \left\| \left\{ \frac{1}{\varepsilon} (T - \alpha) \right\}^\nu x \right\| < +\infty .$$

Proof. Since $\mathfrak{E}_{\alpha,\varepsilon}$ is closed, we have by (16)

$$\varphi(\xi) \in \mathfrak{E}_{\alpha,\varepsilon} \qquad \text{for } \xi \in \mathfrak{V}_{E(\mathfrak{E}_{\alpha,\varepsilon})},$$

that is,

$$\frac{1}{\varepsilon}|\varphi(\xi)-\alpha| \leqq 1 \qquad \text{for } \xi \in \mathfrak{V}_{E(\mathfrak{E}_{\alpha,\varepsilon})}.$$

By virtue of Theorem 22.5, we have

$$\{\frac{1}{\varepsilon}(T-\alpha)\}^{\nu} E(\mathfrak{E}_{\alpha,\varepsilon})x = \int_{E(\mathfrak{E}_{\alpha,\varepsilon})} (\frac{1}{\varepsilon}|\varphi(\xi)-\alpha|)^{\nu} d\xi\, x,$$

because φ is bounded in $\mathfrak{V}_{E(\mathfrak{E}_{\alpha,\varepsilon})}$. Thus we obtain by §17(9)

$$\|\{\frac{1}{\varepsilon}(T-\alpha)\}^{\nu} E(\mathfrak{E}_{\alpha,\varepsilon})x\|^2 = \int_{E(\mathfrak{E}_{\alpha,\varepsilon})} (\frac{1}{\varepsilon}|\varphi(\xi)-\alpha|)^{2\nu} \|d\xi\, x\|^2 \leqq \|E(\mathfrak{E}_{\alpha,\varepsilon})x\|^2,$$

and hence

$$\sup_{\nu \geqq 1} \|\{\frac{1}{\varepsilon}(T-\alpha)\}^{\nu} E(\mathfrak{E}_{\alpha,\varepsilon})x\| \leqq \|E(\mathfrak{E}_{\alpha,\varepsilon})x\|$$

for every $x \in R$.

Conversely we suppose

$$\sup_{\nu \geqq 1} \|\{\frac{1}{\varepsilon}(T-\alpha)\}^{\nu} a\| < +\infty$$

for some $a \in R$. For any $\varepsilon' > \varepsilon$, we have by (16)

$$|\varphi(\xi)-\alpha| \geqq \varepsilon' \qquad \text{for } \xi \in \mathfrak{V}_{E(\mathfrak{E}_{\alpha,\varepsilon'})},$$

and hence by Theorems 22.5 and 17.3

$$\|\{\frac{1}{\varepsilon}(T-\alpha)\}^{\nu} a\|^2 \geqq \|\{\frac{1}{\varepsilon}(T-\alpha)\}^{\nu}(1-E(\mathfrak{E}_{\alpha,\varepsilon'}))a\|^2 = \int_{1-E(\mathfrak{E}_{\alpha,\varepsilon'})} (\frac{1}{\varepsilon}|\varphi(\xi)-\alpha|)^{2\nu} \|d\xi\, a\|^2 \geqq (\frac{\varepsilon'}{\varepsilon})^{2\nu} \|(1-E(\mathfrak{E}_{\alpha,\varepsilon'}))a\|^2$$

for every $\nu = 1, 2, \ldots$. Thus we conclude

$$\|(1-E(\mathfrak{E}_{\alpha,\varepsilon'}))a\| = 0,$$

that is, $a = E(\mathfrak{E}_{\alpha,\varepsilon'})a$ for every $\varepsilon' > \varepsilon$. Since we have by the relation (15)

$$E(\mathfrak{E}_{\alpha,\varepsilon+\frac{1}{\nu}}) \downarrow_{\nu=1}^{\infty} E(\mathfrak{E}_{\alpha,\varepsilon}),$$

we obtain hence $a = E(\mathfrak{E}_{\alpha,\varepsilon})a$.

A bounded operator S on R is said to be commutative to an operator T with domain D, if $ST \subset TS$, that is, if $SD \subset D$ and

$$TSa = STa \qquad \text{for every } a \in D.$$

Theorem 23.3. If a bounded linear operator S on R is com-

mutative to

$$T = \int \varphi(\xi)\, dE_\xi,$$

then for the spectrality E of φ we have

$$SE(\mathfrak{E}) = E(\mathfrak{E})S \qquad \text{for all regularly measurable sets } \mathfrak{E}.$$

Proof. By virtue of Theorem 23.2, putting

$$\mathfrak{E}_{\alpha,\varepsilon} = \{\zeta : |\zeta - \alpha| \leqq \varepsilon\},$$

$E(\mathfrak{E}_{\alpha,\varepsilon})R$ is composed of those elements $x \in R$ for which

$$\sup_{\nu \geqq 1} \| \{\frac{1}{\varepsilon}(T - \alpha)\}^\nu x \| < +\infty.$$

Thus we have by assumption for every $x \in R$

$$\sup_{\nu \geqq 1} \| \{\frac{1}{\varepsilon}(T - \alpha)\}^\nu S E(\mathfrak{E}_{\alpha,\varepsilon}) x \|$$
$$= \sup_{\nu \geqq 1} \| S \{\frac{1}{\varepsilon}(T - \alpha)\}^\nu E(\mathfrak{E}_{\alpha,\varepsilon}) x \| < +\infty$$

because S is bounded by assumption. Therefore we have

$$E(\mathfrak{E}_{\alpha,\varepsilon})S E(\mathfrak{E}_{\alpha,\varepsilon}) = S E(\mathfrak{E}_{\alpha,\varepsilon}).$$

For two measurable sets $\mathfrak{E}$ and $\mathfrak{F}$, if

$$E(\mathfrak{E})S E(\mathfrak{E}) = S E(\mathfrak{E}),$$
$$E(\mathfrak{F})S E(\mathfrak{F}) = S E(\mathfrak{F}),$$

then we have by the formula (17)

$$\begin{aligned} &E(\mathfrak{E} \dotplus \mathfrak{F})S E(\mathfrak{E} \dotplus \mathfrak{F}) \\ &= E(\mathfrak{E})S\{E(\mathfrak{E}) + E(\mathfrak{F}) - E(\mathfrak{E})E(\mathfrak{F})\} \\ &\quad + E(\mathfrak{F})S\{E(\mathfrak{E}) + E(\mathfrak{F}) - E(\mathfrak{E})E(\mathfrak{F})\} \\ &\quad - E(\mathfrak{E})E(\mathfrak{F})S\{E(\mathfrak{E}) + E(\mathfrak{F}) - E(\mathfrak{E})E(\mathfrak{F})\} \\ &= S E(\mathfrak{E}) + E(\mathfrak{E})S E(\mathfrak{F}) - S E(\mathfrak{E})E(\mathfrak{F}) \\ &\quad + E(\mathfrak{F})S E(\mathfrak{E}) + S E(\mathfrak{F}) - S E(\mathfrak{E})E(\mathfrak{F}) \\ &\quad - E(\mathfrak{F})S E(\mathfrak{E}) - E(\mathfrak{E})S E(\mathfrak{F}) + S E(\mathfrak{E})E(\mathfrak{F}) \\ &= S E(\mathfrak{E} \dotplus \mathfrak{F}). \end{aligned}$$

On the other hand, for any complex number α and positive number ε, we can find a sequence of complex numbers α_ν and positive numbers ε_ν $(\nu = 1, 2, \ldots)$ such that

$$\mathfrak{E}_{\alpha,\varepsilon} = \sum_{\nu=1}^{\infty} \mathfrak{E}_{\alpha_\nu,\varepsilon_\nu},$$

and then we have by the formulas (14) and (11)

$$E(\sum_{\nu=1}^{p} \mathfrak{E}_{\alpha_\nu,\varepsilon_\nu}) \uparrow_{p=1}^{\infty} E(\mathfrak{E}_{\alpha,\varepsilon'}) = 1 - E(\mathfrak{E}_{\alpha,\varepsilon}).$$

Therefore we also have

$$(1 - E(\mathfrak{F}_{\alpha,\varepsilon}))S(1 - E(\mathfrak{F}_{\alpha,\varepsilon})) = S(1 - E(\mathfrak{F}_{\alpha,\varepsilon})),$$

and hence

$$E(\mathfrak{F}_{\alpha,\varepsilon})SE(\mathfrak{F}_{\alpha,\varepsilon}) = E(\mathfrak{F}_{\alpha,\varepsilon})S.$$

Consequently we obtain

$$E(\mathfrak{F}_{\alpha,\varepsilon})S = SE(\mathfrak{F}_{\alpha,\varepsilon}).$$

For any open set $\Lambda^\circ \neq 0$ in $\mathfrak{G}$, we can find α_ν, ε_ν ($\nu = 1, 2, \ldots$) such that

$$\Lambda^\circ = \sum_{\nu=1}^{\infty} \mathfrak{F}_{\alpha_\nu, \varepsilon_\nu},$$

and then we have by (14)

$$E(\sum_{\nu=1}^{p} \mathfrak{F}_{\alpha_\nu,\varepsilon_\nu}) \uparrow_{p=1}^{\infty} E(\Lambda^\circ).$$

Thus we conclude easily further

$$E(\Lambda^\circ)S = SE(\Lambda^\circ) \qquad \text{for every open set } \Lambda^\circ.$$

If $\mathfrak{F}$ is regularly measurable, then we have by (18)

$$E(\mathfrak{F}) = \bigwedge_{\mathfrak{F} \subset \Lambda^\circ} E(\Lambda^\circ)$$

For any $\alpha \in R$, we can find by Theorem 14.6 $\ell \in R$ for which

$$C_{[\ell]} = C_{[\alpha]} \vee C_{[\ell\alpha]}.$$

For such $\ell \in R$, we can find by Theorem 14.3 a sequence of open sets $\Lambda_\nu^\circ \supset \mathfrak{F}$ ($\nu = 1, 2, \ldots$) such that

$$\bigwedge_{\nu=1}^{\infty} E(\Lambda_\nu^\circ)[\ell] = E(\mathfrak{F})[\ell],$$

and then we have by the formula §14(16')

$$\bigwedge_{\nu=1}^{\infty} E(\Lambda_\nu^\circ) C_{[\ell]} = E(\mathfrak{F}) C_{[\ell]}.$$

This relation yields by (13)

$$E(\prod_{\nu=1}^{p} \Lambda_\nu^\circ) C_{[\ell]} \downarrow_{p=1}^{\infty} E(\mathfrak{F}) C_{[\ell]}.$$

On the other hand we have by §14(17)

$$C_{[\ell]} Sa = C_{[\ell]} C_{[\ell\alpha]} Sa = C_{[\ell\alpha]} Sa = Sa,$$
$$C_{[\ell]} a = C_{[\ell]} C_{[\alpha]} a = C_{[\alpha]} a = a,$$

and further

$$E(\prod_{\nu=1}^{p} \Lambda_\nu^\circ) Sa = SE(\prod_{\nu=1}^{p} \Lambda_\nu^\circ) a,$$

as proved just above. Therefore we obtain

$$E(\mathfrak{F})Sa = SE(\mathfrak{F})a \qquad \text{for every } a \in R.$$

§24 Spectralisation of Dilatators

In the sequel, we shall prove that for any dilatator T by $\mathcal{R}$, there exists uniquely a continuous function φ on the proper space $\mathcal{E}_{\mathcal{R}}$ of $\mathcal{R}$ such that

$$T = \int \varphi(\xi)\, d\xi .$$

We shall consider firstly the case where $\mathcal{R}$ is an axis, that is, $\mathcal{R} = \overline{\mathcal{R}}$. In this case we have obviously by definition

$$[x] = C_{[x]} \in \mathcal{R} \qquad \text{for every } x \in R .$$

Let T be a dilatator with domain D by an axis $\mathcal{R}$. For every a, $\ell \in D$ we have by Theorems 18.4 and 17.4

$$[a]\ell = [a][\ell]\ell$$
$$= [\ell] \int_{[a]} (\tfrac{\ell}{a}, \xi)\, d\xi\, a = \int_{[a][\ell]} (\tfrac{\ell}{a}, \xi)\, d\xi\, a ,$$

and hence by Theorem 22.1

$$[a]T\ell = T[a]\ell = \int_{[a][\ell]} (\tfrac{\ell}{a}, \xi)\, d\xi\, Ta .$$

Recalling Theorem 18.5, we obtain hence by the formula §19(6)

$$(\tfrac{T\ell}{\ell}, \xi) = (\tfrac{[a]T\ell}{\ell}, \xi) = (\tfrac{\ell}{a}, \xi)\,(\tfrac{Ta}{\ell}, \xi)$$

in an open set being dense in $\mathcal{U}_{[a][\ell]}$. Thus we conclude by the formulas (14) and (6) in §19

$$(\tfrac{T\ell}{\ell}, \xi) = (\tfrac{Ta}{a}, \xi)$$

in an open set being dense in $\mathcal{U}_{[a][\ell]}$. Since both sides are continuous in $\mathcal{U}_{[a][\ell]}$, we obtain hence

$$(\tfrac{T\ell}{\ell}, \xi) = (\tfrac{Ta}{a}, \xi) \qquad \text{for every } \xi \in \mathcal{U}_{[a][\ell]} .$$

On the other hand, $\sum_{x \in D} \mathcal{U}_{[x]}$ is an open dense set in the proper space $\mathcal{E}_{\mathcal{R}}$, because for any $0 \neq P \in \mathcal{R}$ we can find $x \in D$ such that $Px \neq 0$, and then $\mathcal{U}_P\, \mathcal{U}_{[x]} = \mathcal{U}_{[Px]} \neq 0$ by the formula §14(6'). Therefore there exists a continuous function φ on $\mathcal{E}_{\mathcal{R}}$ such that

$$\varphi(\xi) = (\tfrac{Tx}{x}, \xi) \qquad \text{for } \xi \in \mathcal{U}_{[x]},\ x \in D .$$

For such φ, we have by Theorem 18.4

$$Tx = [x]Tx = \int_{[x]} (\tfrac{Tx}{x}, \xi)\, d\xi\, x = \int_{[x]} \varphi(\xi)\, d\xi\, x$$

for every $x \in D$. For an element $a \in R$, if φ is integrable by a in $U_{[a]}$, then we can find by Theorem 22.2 a sequence

$$\mathcal{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} [a]$$

such that $P_\nu a \in D$ for every $\nu = 1, 2, \ldots$, and by Theorem 17.4

$$T P_\nu a = \int_{[P_\nu a]} \varphi(\xi)\, d_\xi P_\nu a = \int_{P_\nu [a]} \varphi(\xi)\, d_\xi a$$

for every $\nu = 1, 2, \ldots$, as proved just above. Since T is closed, we conclude hence by definition $a \in D$ and

$$Ta = \int_{[a]} \varphi(\xi)\, d_\xi a.$$

Therefore we have $T = \int \varphi(\xi)\, d_\xi$ by definition.

Now we consider the general case. Let $\mathcal{R}$ be a closed ring of projection operators containing 1. By virtue of Theorem 20.7, we can find an axis $\mathfrak{A}$ including $\mathcal{R}$. Then we have obviously

$$\mathcal{R} \subset \mathfrak{A} = \overline{\mathfrak{A}} \subset \overline{\mathcal{R}}.$$

Thus every dilatator T by $\mathcal{R}$ also is a dilatator by $\mathfrak{A}$, and hence we can find a continuous function $\tilde{\varphi}$ on the proper space $\mathcal{L}_{\mathfrak{A}}$ of $\mathfrak{A}$ for which we have

$$T = \int \tilde{\varphi}(\xi)\, d\xi,$$

as proved just above. By virtue of Theorem 20.8, the proper space $\mathcal{L}_{\mathcal{R}}$ of $\mathcal{R}$ is a partition space of $\mathcal{L}_{\mathfrak{A}}$ by the correspondence

$$\mathcal{R}\xi \in \mathcal{L}_{\mathcal{R}} \qquad \text{for} \quad \xi \in \mathcal{L}_{\mathfrak{A}},$$

$$U_P^{\mathfrak{A}} = \{ \xi : \mathcal{R}\xi \in U_P^{\mathcal{R}} \} \qquad \text{for} \quad P \in \mathcal{R}.$$

If $\tilde{\varphi}$ is finite at a point $\xi_0 \in \mathcal{L}_{\mathfrak{A}}$, then for any $\varepsilon > 0$, putting

$$\mathfrak{E}_\varepsilon = \{ \zeta : |\zeta - \tilde{\varphi}(\xi_0)| < \varepsilon \},$$

we obtain an open set $\mathfrak{E}_\varepsilon$ in the complex plane G, and then the spectrality $E_{\tilde{\varphi}}(\mathfrak{E}_\varepsilon)$ by $\tilde{\varphi}$ is by Theorem 23.3 commutative to $\overline{\mathcal{R}}$, because T is a dilatator by $\mathcal{R}$ by assumption. Thus we see by Theorem 20.1 that $E_{\tilde{\varphi}}(\mathfrak{E}_\varepsilon) \in \mathcal{R}$ for every $\varepsilon > 0$. Consequently

$$\mathcal{R}\xi = \mathcal{R}\xi_0 \quad \text{implies} \quad \xi \in U^{\mathfrak{A}}_{E_{\tilde{\varphi}}(\mathfrak{E}_\varepsilon)},$$

because we have $\xi_0 \in U^{\mathfrak{A}}_{E_{\tilde{\varphi}}(\mathfrak{E}_\varepsilon)}$ obviously by the formula §23(16). Thus we also see by §23(16) that $\mathcal{R}\xi = \mathcal{R}\xi_0$ implies

$$|\tilde{\varphi}(\xi) - \tilde{\varphi}(\xi_0)| \leqq \varepsilon \qquad \text{for every } \varepsilon > 0.$$

Therefore we obtain that

$$\mathcal{R}\mathfrak{x} = \mathcal{R}\mathfrak{x}_0 \quad \text{implies} \quad \tilde{\varphi}(\mathfrak{x}) = \tilde{\varphi}(\mathfrak{x}_0).$$

If $\tilde{\varphi}(\mathfrak{x}_0) = \infty$, then, putting

$$\mathfrak{E}_\varepsilon = \{ \zeta : \ |\zeta| > \varepsilon \} \qquad \text{for } \varepsilon > 0,$$

we conclude likewise that $\mathcal{R}\mathfrak{x} = \mathcal{R}\mathfrak{x}_0$ implies $\tilde{\varphi}(\mathfrak{x}) = \infty$. Consequently there exists a continuous function φ on the proper space $\mathcal{L}_{\mathcal{R}}$ such that

$$\varphi(\mathcal{R}\mathfrak{x}) = \tilde{\varphi}(\mathfrak{x}) \qquad \text{for every } \mathfrak{x} \in \mathcal{L}_{\mathcal{A}},$$

because $\mathcal{L}_{\mathcal{R}}$ is a partition space of $\mathcal{L}_{\mathcal{A}}$ by the correspondence $\mathcal{R}\mathfrak{x} \in \mathcal{L}_{\mathcal{R}}$ $(\mathfrak{x} \in \mathcal{L}_{\mathcal{A}})$. For such a continuous function φ on $\mathcal{L}_{\mathcal{R}}$, we see easily by the definition of integral and Theorem 17.3 that φ is integrable by χ in $\mathcal{U}_{\mathcal{C}^{\mathcal{R}}_{[\chi]_{\mathcal{R}}}}$ if and only if $\tilde{\varphi}$ is integrable by χ in $\mathcal{U}_{\mathcal{C}^{\mathcal{A}}_{[\chi]_{\mathcal{R}}}}$ and then

$$\int_{\mathcal{C}_{[\chi]_{\mathcal{R}}}} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\chi = \int_{\mathcal{C}_{[\chi]_{\mathcal{R}}}} \tilde{\varphi}(\mathfrak{x})\, d_{\mathfrak{x}}\chi = \int_{[\chi]_{\mathcal{A}}} \tilde{\varphi}(\mathfrak{x})\, d_{\mathfrak{x}}\chi = T\chi,$$

because $\mathcal{C}_{[\chi]_{\mathcal{R}}}\chi = \chi$ by §14(17), and hence by §14(6')

$$[\chi]_{\mathcal{A}} = \mathcal{C}_{[\chi]_{\mathcal{R}}}[\chi]_{\mathcal{A}} \leqq \mathcal{C}_{[\chi]_{\mathcal{R}}}.$$

Therefore we have by definition

$$T = \int \varphi(\mathfrak{x})\, d\mathfrak{x}.$$

Such a continuous function φ on $\mathcal{L}_{\mathcal{R}}$ is determined uniquely. Because for every $\chi \in D$ we have

$$T\chi = \int_{\mathcal{C}_{[\chi]}} \varphi(\mathfrak{x})\, d_{\mathfrak{x}}\chi,$$

and hence by Theorem 18.5 and the formula §19(2)

$$\left(\frac{T\chi}{\chi}, \mathfrak{x}\right) = \varphi(\mathfrak{x}) \qquad \text{for } \mathfrak{x} \in \mathcal{U}_{\mathcal{C}_{[\chi]}}.$$

Furthermore $\sum_{\chi \in D} \mathcal{U}_{\mathcal{C}_{[\chi]}}$ is dense in $\mathcal{L}_{\mathcal{R}}$, because for any $0 \neq \rho \in \mathcal{R}$ we can find $\chi \in D$ such that $\rho\chi \neq 0$, and then we have by §14(16')

$$\mathcal{U}_\rho\, \mathcal{U}_{\mathcal{C}_{[\chi]}} = \mathcal{U}_{\mathcal{C}_{[\rho\chi]}} \neq 0.$$

The function value $\varphi(\mathfrak{x})$ of such a continuous function φ will be called the spectrum of T at a point $\mathfrak{x}$ and denoted by $(T, \mathfrak{x})$.

With this definition we can therefore state

Theorem 24.1. *For a dilatator T with domain D by $\mathcal{R}$, the spectrum $(T, \mathfrak{x})$ $(\mathfrak{x} \in \mathcal{L}_{\mathcal{R}})$ is a continuous function on the proper*

space $\mathcal{C}_{\mathcal{R}}$, and we have

$$T = \int (T, \mathfrak{F})\, d\mathfrak{F},$$

$$\left(\frac{Tx}{x}, \mathfrak{F}\right) = (T, \mathfrak{F}) \quad \text{for } \mathfrak{F} \in \bigcup C_{[x]},\ x \in D.$$

For a dilatator T by $\mathcal{R}$, spectrality E of the spectrum $(T, \mathfrak{F})$ $(\mathfrak{F} \in \mathcal{C}_{\mathcal{R}})$ is called the spectrality of T by $\mathcal{R}$.

A dilatator T by $\mathcal{R}$ also is a dilatator by every closed ring including $\mathcal{R}$, because $\mathcal{R} \subset \mathcal{R}_1$ implies $\overline{\mathcal{R}}_1 \subset \overline{\mathcal{R}}$. Furthermore the spectrality E of T is invariant for regularly measurable sets, that is,

Theorem 24.2. Let T be a dilatator by a closed ring $\mathcal{R}_1$ as well as by $\mathcal{R}_2$. For the spectrality E_1 of T by $\mathcal{R}_1$ and E_2 of T by $\mathcal{R}_2$, a point set $\mathfrak{E}$ in the complex plane G is regularly measurable for E_1 if and only if $\mathfrak{E}$ also is so for E_2, and then

$$E_1(\mathfrak{E}) = E_2(\mathfrak{E}).$$

Proof. By virtue of Theorem 23.2, we have obviously

$$E_1(\mathfrak{E}_{\alpha,\varepsilon}) = E_2(\mathfrak{E}_{\alpha,\varepsilon}) \quad \text{for } \mathfrak{E}_{\alpha,\varepsilon} = \{\zeta : |\zeta - \alpha| \leqq \varepsilon\}$$

For every open set $\Lambda^\circ \neq 0$ in G, we can find α_ν, ε_ν $(\nu = 1, 2, \ldots)$ such that

$$\Lambda^\circ = \sum_{\nu=1}^{\infty} \mathfrak{E}_{\alpha_\nu, \varepsilon_\nu},$$

and then we have by §23(12)

$$E_1(\Lambda^\circ) = \bigcup_{\nu=1}^{\infty} E_1(\mathfrak{E}_{\alpha_\nu,\varepsilon_\nu}) = \bigcup_{\nu=1}^{\infty} E_2(\mathfrak{E}_{\alpha_\nu,\varepsilon_\nu}) = E_2(\Lambda^\circ).$$

Thus we have $E_1(\Lambda^\circ) = E_2(\Lambda^\circ)$ for every open set Λ° in G. Consequently we obtain by §23(11) $E_1(\Lambda^-) = E_2(\Lambda^-)$ for every closed set Λ^- in G. Therefore we have for every set $\mathfrak{E}$ in G

$$\bigwedge_{\mathfrak{E} < \Lambda^\circ} E_1(\Lambda^\circ) = \bigwedge_{\mathfrak{E} < \Lambda^\circ} E_2(\Lambda^\circ),$$

$$\bigvee_{\mathfrak{E} > \Lambda^-} E_1(\Lambda^-) = \bigvee_{\mathfrak{E} > \Lambda^-} E_2(\Lambda^-).$$

Thus we conclude easily our assertion.

Theorem 24.3. Let T be a dilatator with domain D by $\mathcal{R}$ and E the spectrality of T. For a bounded linear operator S on R, we have $ST \subset TS$ if and only if

$$SE(\Lambda^\circ) = E(\Lambda^\circ)S \quad \text{for every open set } \Lambda^\circ.$$

Proof. If $E(\mathfrak{E})S = SE(\mathfrak{E})$, then we have by §23(11)

$$(1 - E(\mathfrak{E}'))S = S(1 - E(\mathfrak{E}')),$$

and hence $E(\mathfrak{E}')S = SE(\mathfrak{E}')$. If

$$E(\mathfrak{E})S = SE(\mathfrak{E}), \qquad E(\mathfrak{F})S = SE(\mathfrak{F}),$$

then we have by the formula §23(17)

$$E(\mathfrak{E} \dot{+} \mathfrak{F})S = (E(\mathfrak{E}) + E(\mathfrak{F}) - E(\mathfrak{E})E(\mathfrak{F}))S$$
$$= S(E(\mathfrak{E}) + E(\mathfrak{F}) - E(\mathfrak{E})E(\mathfrak{F})) = SE(\mathfrak{E} \dot{+} \mathfrak{F}).$$

Furthermore, if $E(\mathfrak{E}_\nu)S = SE(\mathfrak{E}_\nu)$ $(\nu = 1, 2, \ldots)$, $\mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \cdots$, then we also have by the formula §23(14)

$$E(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu)S = SE(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu).$$

Therefore the totality of those point sets $\mathfrak{E}$ for which we have

$$E(\mathfrak{E})S = SE(\mathfrak{E}),$$

constitutes a totally additive set class, and contains all open sets by assumption. Thus we see that $E(\mathfrak{E})S = SE(\mathfrak{E})$ for every Borel set $\mathfrak{E}$ in $\mathfrak{G}$.

For any $\alpha \in D$, we have by Theorem 24.1

$$T\alpha = \int (T, \xi)\, d_\xi \alpha.$$

We can find a sequence of Borel sets $\mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \cdots$ in $\mathfrak{G}$ such that

$$\mathfrak{G} = \sum_{\nu=1}^{\infty} \mathfrak{E}_\nu, \qquad \sup_{\zeta \in \mathfrak{E}_\nu} |\zeta| \leqq \nu \quad (\nu = 1, 2, \ldots).$$

Then we have $E(\mathfrak{E}_\nu) \uparrow_{\nu=1}^{\infty} 1$ by §23(14), and

$$|(T, \xi)| \leqq \nu \qquad \text{for } \xi \in \mathfrak{U}_{E(\mathfrak{E}_\nu)}$$

by §23(16), because $\mathfrak{E}_\nu^- \subset \{\zeta : |\zeta| \leqq \nu\}$. Thus we have by Theorem 17.4

$$E(\mathfrak{E}_\nu)T\alpha = \int_{E(\mathfrak{E}_\nu)} (T, \xi)\, d_\xi \alpha \quad (\nu = 1, 2, \ldots).$$

For any $\varepsilon > 0$, we can find a finite number of Borel sets $\mathfrak{F}_{\nu,\mu}$ $(\mu = 1, 2, \ldots, \rho)$ in $\mathfrak{G}$ such that

$$\mathfrak{E}_\nu = \sum_{\mu=1}^{\rho} \mathfrak{F}_{\nu,\mu}, \qquad \mathfrak{F}_{\nu,\mu}\mathfrak{F}_{\nu,\sigma} = 0 \quad \text{for } \mu \neq \sigma,$$

$$\sup_{\xi, \eta \in \mathfrak{F}_{\nu,\mu}} |\xi - \eta| \leqq \varepsilon \quad \text{for } \mu = 1, 2, \ldots, \rho.$$

Then we have by the formulas (7) and (12) in §23

$$E(\mathfrak{E}_\nu) = \sum_{\mu=1}^{\rho} E(\mathfrak{F}_{\nu,\mu}),$$

and by the formula §23(16)

$$|(T,\xi)-(T,\eta)| \leqq \varepsilon \qquad \text{for } \xi,\ \eta \in \mathfrak{V}_{E(\Psi_{\nu,\mu})}$$

because $\Psi_{\nu,\mu}^{-} \subset \{\zeta : |\zeta-(T,\eta)| \leqq \varepsilon\}$ for $\eta \in \mathfrak{V}_{E(\Psi_{\nu,\mu})}$. Thus we have by the definition of integral

$$\int_{E(\mathfrak{X}_\nu)} (T,\xi)\,d_\xi a = \lim_{\varepsilon\to 0} \sum_{\mu=1}^{\rho} (T,\xi_\mu) E(\Psi_{\nu,\mu})a$$

for $\xi_\mu \in \mathfrak{V}_{E(\Psi_{\nu,\mu})}$ $(\mu = 1, 2, \dots, \rho)$, and hence

$$E(\mathfrak{X}_\nu)STa = SE(\mathfrak{X}_\nu)Ta = S\int_{E(\mathfrak{X}_\nu)} (T,\xi)\,d_\xi a$$

$$= \lim_{\varepsilon\to 0} \sum_{\mu=1}^{\rho} (T,\xi_\mu)E(\Psi_{\nu,\mu})Sa = \int_{E(\mathfrak{X}_\nu)} (T,\xi)\,d_\xi Sa$$

for every $\nu = 1, 2, \dots$. This relation yields

$$STa = \lim_{\nu\to\infty} \int_{E(\mathfrak{X}_\nu)} (T,\xi)\,d_\xi Sa,$$

and further by §17(9)

$$\int_{E(\mathfrak{X}_\nu)} |(T,\xi)|^2 \|d_\xi Sa\|^2 = \|E(\mathfrak{X}_\nu)STa\|^2 \leqq \|STa\|^2$$

for every $\nu = 1, 2, \dots$. Therefore (T,ξ) is by Theorem 17.3 integrable by Sa in $\mathfrak{L}_{\mathfrak{R}}$ and

$$STa = \int (T,\xi)\,d_\xi Sa = TSa.$$

Therefore S is commutative to T by definition. Conversely, if we have $ST \subset TS$, then we obtain by Theorem 23.3

$$SE(\mathfrak{X}) = E(\mathfrak{X})S \qquad \text{for every Borel set } \mathfrak{X} \text{ in } G.$$

In this Proof, putting $P_\nu = E(\mathfrak{X}_\nu)$ $(\nu = 1, 2, \dots)$, we have

$$P_\nu \uparrow_{\nu=1}^{\infty} 1, \qquad |(T,\xi)| \leqq \nu \qquad \text{for } \xi \in \mathfrak{V}_{P_\nu}.$$

Thus (T,ξ) is integrable by $P_\nu x$ in $\mathfrak{L}_{\mathfrak{R}}$ for every $x \in R$; and

$$\|TP_\nu x\|^2 = \int_{P_\nu} |(T,\xi)|^2 \|d_\xi x\|^2 \leqq \nu^2 \|x\|^2.$$

Therefore we have

Theorem 24.4. For a dilatator T with domain D there exists a sequence $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} 1$ such that $P_\nu R \subset D$ and TP_ν is a bounded linear operator on $P_\nu R$ for every $\nu = 1, 2, \dots$.

Let T be a dilatator by $\mathfrak{R}$. For the spectrality E of T, the

closed ring generated by $E(\Lambda^\circ)$ for all open sets Λ° in $\mathfrak{G}$ is called the spectral ring of T. With this definition, we see obviously by Theorem 24.2 that the spectral ring of T is independent from $\mathfrak{R}$. Furthermore we see at once by Theorem 24.3 that every dilatator by $\mathfrak{R}$ also is a dilatator by its spectral ring.

A dilatator T by $\mathfrak{R}$ is said to be proper, if $\mathfrak{R}$ coincides with the spectral ring of T.

Theorem 24.5. For the spectrality E of a dilatator T by $\mathfrak{R}$, if a projection operator P belongs to the spectral ring of T, then for any $\alpha \in R$ we can find a Borel set $\overset{*}{\mathfrak{E}}$ in $\mathfrak{G}$ such that

$$P\,C_{[\alpha]} = E(\overset{*}{\mathfrak{E}})\,C_{[\alpha]}.$$

Proof. We denote by $\mathfrak{r}$ the totality of those projection operatore $P \in \mathfrak{R}$, for which we can find Borel sets $\overset{*}{\mathfrak{E}}$ such that

$$P\,C_{[\alpha]} = E(\overset{*}{\mathfrak{E}})\,C_{[\alpha]}.$$

Then we see easily by the formulas (11) and (12) in §23 that $\mathfrak{r}$ is a ring including all $E(\Lambda^\circ)$ for open sets Λ° in $\mathfrak{G}$. For any system $P_\lambda \in \mathfrak{r}$ $(\lambda \in \Lambda)$, we can find by Theorem 14.7 a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ for which

$$\bigcup_{\lambda\in\Lambda} P_\lambda\,C_{[\alpha]} = \bigcup_{\nu=1}^{\infty} P_{\lambda_\nu}\,C_{[\alpha]},$$

$$\bigcap_{\lambda\in\Lambda} P_\lambda\,C_{[\alpha]} = \bigcap_{\nu=1}^{\infty} P_{\lambda_\nu}\,C_{[\alpha]}.$$

Since we can find by assumption a sequence of Borel sets $\overset{*}{\mathfrak{E}}_\nu$ $(\nu = 1, 2, \ldots)$ for which $P_{\lambda_\nu}\,C_{[\alpha]} = E(\overset{*}{\mathfrak{E}}_\nu)\,C_{[\alpha]}$ $(\nu = 1, 2, \ldots)$, we have by the formulas (12) and (13) in §23 for such $\overset{*}{\mathfrak{E}}_\nu$ $(\nu = 1, 2, \ldots)$

$$\bigcup_{\lambda\in\Lambda} P_\lambda\,C_{[\alpha]} = E\Big(\sum_{\nu=1}^{\infty} \overset{*}{\mathfrak{E}}_\nu\Big)\,C_{[\alpha]},$$

$$\bigcap_{\lambda\in\Lambda} P_\lambda\,C_{[\alpha]} = E\Big(\prod_{\nu=1}^{\infty} \overset{*}{\mathfrak{E}}_\nu\Big)\,C_{[\alpha]},$$

and hence $\bigcup_{\lambda\in\Lambda} P_\lambda$, $\bigcap_{\lambda\in\Lambda} P_\lambda \in \mathfrak{r}$. Thus $\mathfrak{r}$ is closed. Therefore $\mathfrak{r}$ includes the spectral ring of T.

For two closed rings of projection operators $1 \in \mathfrak{r} \subset \mathfrak{R}$, if T is a dilatator by $\mathfrak{r}$, then T also is obviously a dilatator by $\mathfrak{R}$. For the partition correspondence §20(4): $\mathfrak{r}\mathfrak{F} \in \mathfrak{L}_\mathfrak{r}$ $(\mathfrak{F} \in \mathfrak{L}_\mathfrak{R})$ from the proper space $\mathfrak{L}_\mathfrak{R}$ of $\mathfrak{R}$ onto the proper space $\mathfrak{L}_\mathfrak{r}$ of $\mathfrak{r}$, putting

$$\varphi(\mathfrak{F}) = (T, \mathfrak{r}\mathfrak{F})_\mathfrak{r} \qquad (\mathfrak{F} \in \mathfrak{L}_\mathfrak{R})$$

for the spectrum $(T, q)_{\mathfrak{r}}$ $(q \in \mathcal{E}_{\mathfrak{r}})$ by $\mathfrak{r}$, we obtain a continuous function φ on $\mathcal{E}_{\mathfrak{R}}$. Furthermore we see easily by §20(6) that φ is finite in some open dense set. Thus, putting

$$T_1 = \int \varphi(\mathfrak{x})\, d\mathfrak{x},$$

we obtain by Theorem 22.3 a dilatator T_1 by $\mathfrak{R}$. Then we see easily by the definition of integral that $T = T_1$, and hence

$$(T, \mathfrak{x})_{\mathfrak{R}} = \varphi(\mathfrak{x}) \qquad \text{for every } \mathfrak{x} \in \mathcal{E}_{\mathfrak{R}}.$$

Therefore we can state

Theorem 24.6. *For two closed rings of projection operators*

$$1 \in \mathfrak{r} \subset \mathfrak{R},$$

if T *is a dilatator by* $\mathfrak{r}$, *then for the spectrum* $(T, q)_{\mathfrak{r}}$ *by* $\mathfrak{r}$ *and the spectrum* $(T, \mathfrak{x})_{\mathfrak{R}}$ *by* $\mathfrak{R}$ *we have*

$$(T, \mathfrak{r}\mathfrak{x})_{\mathfrak{r}} = (T, \mathfrak{x})_{\mathfrak{R}} \qquad (\mathfrak{x} \in \mathcal{E}_{\mathfrak{R}})$$

for the partition correspondence $\mathfrak{r}\mathfrak{x} \in \mathcal{E}_{\mathfrak{r}}$ $(\mathfrak{x} \in \mathcal{E}_{\mathfrak{R}})$ *from the proper space* $\mathcal{E}_{\mathfrak{R}}$ *of* $\mathfrak{R}$ *onto the proper space* $\mathcal{E}_{\mathfrak{r}}$ *of* $\mathfrak{r}$.

§25 Calculus of Dilatators

Let T_1 and T_2 be two dilatators by $\mathfrak{R}$ with domain D_1 and D_2 respectively. Since the spectrum of a dilatator by $\mathfrak{R}$ is finite continuous in some open dense set in the proper space $\mathcal{E}_{\mathfrak{R}}$, putting

$$A = \{\mathfrak{x} : |(T_1, \mathfrak{x})| < +\infty, \quad |(T_2, \mathfrak{x})| < +\infty\},$$

we see easily that A also is an open dense set. Since

$$(T_1, \mathfrak{x}) + (T_2, \mathfrak{x})$$

is finite continuous in A, we obtain by Theorem 22.3 a dilatator T_3 by $\mathfrak{R}$ as

$$T_3 = \int \{(T_1, \mathfrak{x}) + (T_2, \mathfrak{x})\}\, d\mathfrak{x}.$$

This dilatator T_3 will be denoted by $T_1 \oplus T_2$, that is,

$$(1) \qquad T_1 \oplus T_2 = \int \{(T_1, \mathfrak{x}) + (T_2, \mathfrak{x})\}\, d\mathfrak{x}.$$

With this definition we have obviously

$$(2) \qquad T_1 \oplus T_2 = T_2 \oplus T_1,$$

$$(3) \qquad (T_1 \oplus T_2) \oplus T_3 = T_1 \oplus (T_2 \oplus T_3).$$

We also define likewise

$$(4) \qquad T_1 \circ T_2 = \int (T_1, \mathfrak{x})(T_2, \mathfrak{x})\, d\mathfrak{x},$$

and we have

$$(5) \qquad T_1 \circ T_2 = T_2 \circ T_1,$$

$$(6) \qquad (T_1 \circ T_2) \circ T_3 = T_1 \circ (T_2 \circ T_3),$$

$$(7) \qquad T_1 \circ (T_2 \oplus T_3) = (T_1 \circ T_2) \oplus (T_1 \circ T_3).$$

Recalling Theorem 18.5, we have obviously by definition

$$(8) \qquad (T_1 \oplus T_2, \mathfrak{x}) = (T_1, \mathfrak{x}) + (T_2, \mathfrak{x}),$$

$$(9) \qquad (T_1 \circ T_2, \mathfrak{x}) = (T_1, \mathfrak{x})(T_2, \mathfrak{x}),$$

if both $(T_1, \mathfrak{x})$ and $(T_2, \mathfrak{x})$ are finite.

For two operators T_1 and T_2 with domain D_1 and D_2 respectively, we define $T_1 \subset T_2$ to mean that $D_1 \subset D_2$ and

$$T_1 \chi = T_2 \chi \qquad \text{for every } \chi \in D_1.$$

With this definition we have for two dilatators T_1 and T_2 by $\mathcal{R}$

$$(10) \qquad T_1 + T_2 \subset T_1 \oplus T_2.$$

Because, if both $(T_1, \mathfrak{x})$ and $(T_2, \mathfrak{x})$ are integrable by α in $\bigcup c_{[\alpha]}$, then $(T_1, \mathfrak{x}) + (T_2, \mathfrak{x})$ also is by Theorem 17.6 integrable by α in $\bigcup c_{[\alpha]}$ and

$$\int_{c_{[\alpha]}} \{(T_1, \mathfrak{x}) + (T_2, \mathfrak{x})\}\, d\mathfrak{x}\alpha = \int_{c_{[\alpha]}} (T_1, \mathfrak{x})\, d\mathfrak{x}\alpha + \int_{c_{[\alpha]}} (T_2, \mathfrak{x})\, d\mathfrak{x}\alpha.$$

Thus we have $T_1 + T_2 \subset T_1 \oplus T_2$ by definition.

We have further

$$(11) \qquad T_1 T_2 \subset T_1 \circ T_2.$$

In fact, if $T_1 T_2 \alpha$ has any sense, then we have by Theorem 24.1

$$T_1 T_2 \alpha = \int_{c_{[T_2\alpha]}} (T_1, \mathfrak{x})\, d\mathfrak{x}\, T_2\alpha = \int_{c_{[\alpha]}} (T_1, \mathfrak{x})\, d\mathfrak{x}\, T_2\alpha,$$

because we have by the formula §14(16')

$$c_{[T_2\alpha]} = c_{[T_2 c_{[\alpha]}\alpha]} = c_{[\alpha]} c_{[T_2\alpha]} \leqq c_{[\alpha]}.$$

Thus we obtain by Theorem 18.5

$$\left(\frac{T_1 T_2 a}{a}, \mathfrak{f}\right) = (T_1, \mathfrak{f})\left(\frac{T_2 a}{a}, \mathfrak{f}\right) = (T_1, \mathfrak{f})(T_2, \mathfrak{f})$$

in some open set being dense in $\mathfrak{C}_{[a]}$. This relation yields by Theorem 18.4

$$T_1 T_2 a = [a] T_1 T_2 a = \int_{\mathfrak{C}_{[a]}} (T_1, \mathfrak{f})(T_2, \mathfrak{f})\, d\mathfrak{f} a = (T_1 \circ T_2) a .$$

Therefore we have $T_1 T_2 \subset T_1 \circ T_2$ by definition.

Concerning the adjoint operator T^* of a dilatator T by $\mathfrak{R}$, we have by Theorem 22.4

$$(12) \qquad (T^*, \mathfrak{f}) = \overline{(T, \mathfrak{f})} \qquad \text{for every } \mathfrak{f} \in \mathfrak{L}_{\mathfrak{R}}.$$

Thus we conclude easily

$$(13) \qquad (T^*)^* = T,$$

and further by the formulas (8) and (9)

$$(14) \qquad (T_1 \oplus T_2)^* = T_1^* \oplus T_2^*,$$

$$(15) \qquad (T_1 \circ T_2)^* = T_1^* \circ T_2^*.$$

Recalling Theorem 17.9, we have for every x, $y \in D$

$$(Tx, Ty) = \int (T, \mathfrak{f})\,\overline{(T, \mathfrak{f})}\,(d\mathfrak{f} x, y) = (T^* x, T^* y).$$

Thus we obtain

$$(16) \qquad (Tx, Ty) = (T^* x, T^* y) \qquad (x, y \in D),$$

$$(17) \qquad \|Tx\| = \|T^* x\| \qquad (x \in D).$$

Every complex number α may be considered itself as a dilatator by $\mathfrak{R}$, and we have obviously by definition

$$\alpha^* = \bar{\alpha} \qquad \text{for every complex number } \alpha .$$

Furthermore it is evident by definition that

$$\alpha T = \alpha \circ T,$$

considering α as a dilatator by $\mathfrak{R}$.

A dilatator T by $\mathfrak{R}$ with domain D is said to be <u>symmetric</u>, if $T = T^*$, that is, if

$$(Tx, y) = (x, Ty) \qquad \text{for every } x, y \in D .$$

With this definition we have obviously by the formula (12)

<u>Theorem</u> 25.1. <u>A dilatator</u> T <u>by</u> $\mathfrak{R}$ <u>is symmetric, if and only</u>

if the spectrum $(T, \mathfrak{x})$ is a real valued function over the proper space $\mathfrak{E}_{\mathfrak{R}}$.

For two symmetric dilatators T_1 and T_2 by $\mathfrak{R}$, we define $T_1 \leqq T_2$ or $T_2 \geqq T_1$ to mean that

$$(T_1, \mathfrak{x}) \leqq (T_2, \mathfrak{x}) \qquad \text{for every } \mathfrak{x} \in \mathfrak{E}_{\mathfrak{R}}.$$

With this definition, we have obviously for symmetric dilatators

(18) $T_1 \leqq T_2$ and $T_2 \leqq T_1$ if and only if $T_1 = T_2$,

(19) $T_1 \leqq T_2$, $T_2 \leqq T_3$ implies $T_1 \leqq T_3$,

(20) $T_1 \leqq T_2$ implies $T_1 \oplus T_3 \leqq T_2 \oplus T_3$,

(21) $T_1 \leqq T_2$, $0 \leqq T_3$ implies $T_1 \circ T_3 \leqq T_2 \circ T_3$.

A dilatator T by $\mathfrak{R}$ is said to be positive, if $T \geqq 0$, and to be negative, if $T \leqq 0$.

Theorem 25.2. For two symmetric dilatators T_1 and T_2 with domain D_1 and D_2 respectively, we have $T_1 \leqq T_2$ if and only if

$$(T_1 x, x) \leqq (T_2 x, x) \qquad \text{for every } x \in D_1 D_2.$$

Proof. If $T_1 \leqq T_2$, then we have by definition for every element $x \in D_1 D_2$

$$(T_1 x, x) = \int_{\mathfrak{C}(x)} (T_1, \mathfrak{x}) \| d\mathfrak{x}\, x \|^2 \leqq \int_{\mathfrak{C}(x)} (T_2, \mathfrak{x}) \| d\mathfrak{x}\, x \|^2 = (T_2 x, x),$$

recalling Theorem 17.10. Conversely, if

$$(T_1 x, x) \leqq (T_2 x, x) \qquad \text{for every } x \in D_1 D_2,$$

then for T_1, T_2, we can find by Theorem 24.4 two sequences

$$\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} 1, \qquad \mathfrak{R} \ni Q_\nu \uparrow_{\nu=1}^{\infty} 1,$$

such that $P_\nu R \subset D_1$ and $Q_\nu R \subset D_2$ for every $\nu = 1, 2, \ldots$, and then we have by Theorem 13.8

$$\mathfrak{R} \ni P_\nu Q_\nu \uparrow_{\nu=1}^{\infty} 1$$

and further obviously

$$P_\nu Q_\nu R \subset D_1 D_2 \qquad \text{for every } \nu = 1, 2, \ldots .$$

For any $x \in R$ and $\mathfrak{x} \in \cup_{P_\nu Q_\nu} \mathfrak{C}(x)$, we have by definition

$$(T_1, \mathfrak{x}) = \left(\frac{T_1 P_\nu Q_\nu x}{P_\nu Q_\nu x}, \mathfrak{x} \right) = \lim_{P \to \mathfrak{x}} \frac{(T_1 P P_\nu Q_\nu x, x)}{\| P P_\nu Q_\nu x \|^2}$$

$$\leqq \lim_{P \to \mathfrak{F}} \frac{(T_2 P P_\nu Q_\nu x, x)}{\| P P_\nu Q_\nu x \|^2} = (\frac{T_2 P_\nu Q_\nu x}{P_\nu Q_\nu x}, \mathfrak{F}) = (T_2, \mathfrak{F}).$$

On the other hand we see by Theorems 15.7 and 14.1 that both $\sum_{\nu=1}^{\infty} U_{P_\nu Q_\nu}$ and $\sum_{x \in R} U_{\mathfrak{C}[x]}$ are open dense sets in $\mathfrak{C}_{\mathfrak{R}}$. Thus we conclude

$$(T_1, \mathfrak{F}) \leqq (T_2, \mathfrak{F}) \qquad \text{for every } \mathfrak{F} \in \mathfrak{C}_{\mathfrak{R}},$$

because both sides are continuous functions over the whole space $\mathfrak{C}_{\mathfrak{R}}$.

For a dilatator T by $\mathfrak{R}$ we define the <u>absolute operator</u> $|T|$ of T to mean

$$|T| = \int |(T, \mathfrak{F})| \, d\mathfrak{F}. \tag{22}$$

With this definition, the absolute operator $|T|$ of a dilatator T is obviously a positive dilatator, and

$$(|T|, \mathfrak{F}) = |(T, \mathfrak{F})| \qquad \text{for every } \mathfrak{F} \in \mathfrak{C}_{\mathfrak{R}}. \tag{23}$$

Furthermore we have by the formula (12)

$$T \circ T^* = |T| \circ |T|, \quad |T^*| = |T|. \tag{24}$$

<u>Theorem</u> 25.3. <u>For two dilatators</u> T_1 <u>and</u> T_2 <u>by</u> $\mathfrak{R}$ <u>with domain</u> D_1 <u>and</u> D_2 <u>respectively, we have</u> $|T_1| \leqq |T_2|$ <u>if and only if</u> $D_1 \supset D_2$ <u>and</u> $\| T_1 x \| \leqq \| T_2 x \|$ <u>for every</u> $x \in D_2$.

<u>Proof</u>. If $|T_1| \leqq |T_2|$, then we have by definition

$$|(T_1, \mathfrak{F})| \leqq |(T_2, \mathfrak{F})| \qquad \text{for every } \mathfrak{F} \in \mathfrak{C}_{\mathfrak{R}}.$$

Recalling Theorem 17.3, we conclude hence that $D_1 \supset D_2$ and for every $x \in D_2$ we have

$$\| T_1 x \|^2 = \int_{\mathfrak{C}[x]} |(T_1, \mathfrak{F})| \, \| d\mathfrak{F} \, x \|^2 \leqq \int_{\mathfrak{C}[x]} |(T_2, \mathfrak{F})| \, \| d\mathfrak{F} \, x \|^2 = \| T_2 x \|^2.$$

Conversely, if $\| T_1 x \| \leqq \| T_2 x \|$ for every $x \in D_1 D_2$, then we can find by Theorem 24.4 a sequence $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} 1$ such that

$$P_\nu R \subset D_1 D_2 \qquad \text{for every } \nu = 1, 2, \ldots,$$

and then for any $x \in R$ and $\mathfrak{F} \in U_{P_\nu} \mathfrak{C}_{[x]}$ we have by §19(12)

$$|(T_1, \mathfrak{F})| = |(\frac{T_1 P_\nu x}{P_\nu x}, \mathfrak{F})| = \lim_{P \to \mathfrak{F}} \frac{\| T_1 P P_\nu x \|}{\| P P_\nu x \|} \leqq \lim_{P \to \mathfrak{F}} \frac{\| T_2 P P_\nu x \|}{\| P P_\nu x \|} = |(T_2, \mathfrak{F})|.$$

Thus we conclude easily that $|T_1| \leqq |T_2|$.

From this Theorem 25.3 we conclude immediately

$$\| |T| x \| = \| T x \| \qquad \text{for every } x \in D. \tag{25}$$

If $T \geqq 0$, then we have obviously $T = |T|$ by definition. For an arbitrary dilatator T by $\mathfrak{R}$, putting

$$\sqrt{|T|} = \int \sqrt{|(T, \xi)|}\, d\xi, \tag{26}$$

we obtain a positive dilatator $\sqrt{|T|}$, and we have by definition

$$|T| = \sqrt{|T|} \circ \sqrt{|T|} = \sqrt{|T|}\,\sqrt{|T|}. \tag{27}$$

Because, if $|(T, \xi)|$ is integrable by α in $\mathfrak{U}_{\mathfrak{C}_{[\alpha]}}$, then $\sqrt{|(T, \xi)|}$ also is integrable by α in $\mathfrak{U}_{\mathfrak{C}_{[\alpha]}}$. By virtue of Theorem 24.4, we can find a sequence $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} 1$ such that the domain of $\sqrt{|T|}$ includes $P_\nu R$ for every $\nu = 1, 2, \ldots$, and then we have

$$P_\nu |T| \alpha = \sqrt{|T|}\, P_\nu \sqrt{|T|}\, \alpha \qquad \text{for every } \nu = 1, 2, \ldots .$$

This relation yields $|T| \alpha = \sqrt{|T|}\,\sqrt{|T|}\,\alpha$, and hence

$$|T| \subset \sqrt{|T|}\,\sqrt{|T|}.$$

Thus we obtain $|T| = \sqrt{|T|}\,\sqrt{|T|}$ by the formula (11).

For a dilatator T by $\mathfrak{R}$ with domain D, the projection operator of the closed linear manifold generated by the <u>range</u> of T:

$$\{ T x : x \in D \}$$

is called the <u>projector</u> of T and denoted by $[T]$.

Since $x \in D$, $Q \in \overline{\mathfrak{R}}$ implies $Q x \in D$, $TQx = QTx$, we see by Theorems 10.5 and 12.6 that $Q[T] = [T]Q$ for every $Q \in \overline{\mathfrak{R}}$, and hence we obtain by Theorem 20.1

$$[T] \in \mathfrak{R} \qquad \text{for every dilatator } T \text{ by } \mathfrak{R}. \tag{28}$$

Since $PTx = TPx$ for every $P \in \mathfrak{R}$, $x \in D$, we conclude easily by definition

$$[T] = \bigcup_{x \in D} [Tx]. \tag{29}$$

Since $\mathfrak{C}_{[T]} = [T]$ by (28), we obtain hence by §14(15)

$$[T] = \bigcup_{x \in D} \mathfrak{C}_{[Tx]}. \tag{30}$$

By virtue of Theorem 24.4, we can find a sequence $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} 1$ such that $P_\nu R \subset D$ for every $\nu = 1, 2, \ldots$, and then we have by defi-

nition for every $x \in R$

$$T(1-[T])P_\nu x = (1-[T])TP_\nu x = 0.$$

Since $\lim_{\nu\to\infty}(1-[T])P_\nu x = (1-[T])x$ and T is closed, we obtain hence

$$T(1-[T])x = 0 \qquad \text{for every } x \in R.$$

Conversely, if $Ta = 0$, then we have by (17) $T^*a = 0$, and hence

$$(Tx, a) = (x, T^*a) = 0 \qquad \text{for every } x \in D.$$

Consequently we obtain by definition $[T]a = 0$, that is,

$$(1-[T])a = a \qquad \text{if } Ta = 0.$$

Therefore we have

(31) $$(1-[T])R = \{x : Tx = 0\}.$$

From this formula (31) we conclude by the formulas (17) and (25)

(32) $$[T^*] = [|T|] = [T].$$

If $[T]x \in D$, then we have by (31)

$$x = (1-[T])x + [T]x \in D,$$
$$Tx = T[T]x.$$

Thus we have

(33) $$T[T] = T.$$

Recalling the formula §14(2), we obtain by (29)

(34) $$[\alpha T] = [T] \quad \text{for every complex number } \alpha \neq 0.$$

For every $x \in D$ we have by (33)

$$[T]Tx = T[T]x = Tx,$$

and hence by §14(6')

$$[Tx] = [T][Tx] \leqq [T].$$

We also have by §14(3)

$$[x]Tx = T[x]x = Tx,$$

and hence $[Tx] \leqq [x]$ by §14(1). Thus we obtain

$$[Tx] \leqq [T][x].$$

On the other hand, we have by §14(3) for every $x \in R$

$$T(1-[Tx])x = (1-[Tx])Tx = 0,$$

and hence by (31)

$$[T](1-[Tx])x = 0.$$

This relation yields by §14(6)

$$[T](1-[Tx])[x]=0,$$

because $[x][Tx]=[Tx][x]=[Tx]$. Thus we have

$$[T][x]=[T][Tx][x]\leqq[Tx].$$

Consequently we obtain

(35) $\quad [Tx]=[T][x] \quad$ for every $x\in D$.

Recalling Theorem 15.7, we have by (30)

$$U_{[T]}=\Big(\sum_{x\in D} U_{C_{[Tx]}}\Big)^{-}.$$

On the other hand we have by Theorem 19.2

$$U_{C_{[Tx]}}=\{\xi:\ (\tfrac{Tx}{x},\xi)\neq 0\}^{-},$$

and it is evident by Theorem 24.1 that

$$\{\xi:\ (T,\xi)\neq 0\}^{-}=\Big(\sum_{x\in D}\{\xi:\ (\tfrac{Tx}{x},\xi)\neq 0\}^{-}\Big)^{-}.$$

Thus we obtain

(36) $\quad U_{[T]}=\{\xi:\ (T,\xi)\neq 0\}^{-}.$

By virtue of This formula (36), we obtain by (9) for every pair of dilatators T_1 and T_2 by $\mathfrak{R}$

$$U_{[T_1\circ T_2]}=U_{[T_1]}U_{[T_2]},$$

and hence by the formulas (2) and (4) in §15

(37) $\quad [T_1\circ T_2]=[T_1][T_2].$

Furthermore we obtain by (36)

(38) $\quad |T_1|\leqq|T_2|$ implies $[T_1]\leqq[T_2]$.

For a dilatator $T\neq 0$ by $\mathfrak{R}$, we have obviously $[T]\neq 0$ by definition. Putting

$$\varphi(\xi)=\begin{cases}\dfrac{1}{(T,\xi)} & \text{for } \xi\in U_{[T]}\\ 0 & \text{for } \xi\in U_{1-[T]},\end{cases}$$

we see easily by (36) that φ is almost continuous in the proper space $\mathfrak{L}_{\mathfrak{R}}$, and hence we obtain by Theorem 22.3 a dilatator

$$T_1=\int\varphi(\xi)\,d\xi.$$

This dilatator T_1 by $\mathfrak{R}$ is called the <u>inverse</u> of a dilatator T and denoted by T^{-1}. With this definition, we have by Theorem 24.1

$$(39)\qquad (T^{-1},\mathfrak{F}) = \frac{1}{(T,\mathfrak{F})} \qquad \text{for } \mathfrak{F} \in \mathfrak{U}_{[T]},$$

and hence by (4)

$$(40)\qquad T \circ T^{-1} = [T].$$

We also obtain by (36) and (39)

$$(41)\qquad [T^{-1}] = [T],$$

$$(42)\qquad (T^{-1})^{-1} = T,$$

and further by (12) and (23)

$$(43)\qquad (T^{*})^{-1} = (T^{-1})^{*},$$

$$(44)\qquad |T|^{-1} = |T^{-1}|.$$

If $a = T\mathfrak{l}$, then we have by §19(15) for every $\mathfrak{F} \in \mathfrak{U}_{\mathfrak{C}_{[\mathfrak{l}]a]}}$

$$\left(\frac{\mathfrak{l}}{[\mathfrak{l}]a},\mathfrak{F}\right) = \frac{1}{\left(\frac{a}{\mathfrak{F}},\mathfrak{F}\right)} = \frac{1}{(T,\mathfrak{F})} = (T^{-1},\mathfrak{F}).$$

Since $[\mathfrak{l}]a = T[\mathfrak{l}]\mathfrak{l} = T\mathfrak{l} = a$, we obtain hence by §19(10)

$$[a]\mathfrak{l} = \int_{\mathfrak{C}_{[a]}} (T^{-1},\mathfrak{F})\, d\mathfrak{F}\, a = T^{-1}a.$$

On the other hand we have by (35)

$$[a]\mathfrak{l} = [T\mathfrak{l}]\mathfrak{l} = [T][\mathfrak{l}]\mathfrak{l} = [T]\mathfrak{l}.$$

Therefore we obtain

$$(45)\qquad a = T\mathfrak{l} \text{ implies } [T]\mathfrak{l} = T^{-1}a.$$

If a dilatator T by $\mathfrak{R}$ with domain D is bounded and

$$\|Tx\| \leqq \gamma \|x\| \qquad \text{for every } x \in D,$$

then we have by the formula §19(12)

$$\left|\left(\frac{Tx}{x},\mathfrak{F}\right)\right| \leqq \gamma \qquad \text{for } \mathfrak{F} \in \mathfrak{U}_{\mathfrak{C}_{[x]}},$$

and hence by Theorem 24.1

$$|(T,\mathfrak{F})| \leqq \gamma \qquad \text{for every } \mathfrak{F} \in \mathfrak{F}_{\mathfrak{R}}.$$

This relation yields $|T| \leqq \gamma$ by definition, considering γ as a dilatator by $\mathfrak{R}$. Conversely, if $|T| \leqq \gamma$, then we have by Theorem 25.3 that $D = R$ and

$$\|Tx\| \leqq \gamma \|x\| \qquad \text{for every } x \in R.$$

Therefore we obtain

Theorem 25.4. A dilatator T by $\mathfrak{R}$ is bounded if and only if $|T| \leqq \gamma$ for some positive number γ, and then we have

$$\|T\| = \inf_{|T| \leqq \xi} \xi, \qquad |T| \leqq \|T\|.$$

The domain of a bounded dilatator coincides with the whole space R, as proved just above. Conversely, if the domain of a dilatator T by $\mathfrak{R}$ is the whole space R, then the domain of the adjoint operator T^* also coincides with R by (12), and hence we have for every $y \in R$

$$\sup_{\|x\| \leqq 1} |(Tx, y)| = \sup_{\|x\| \leqq 1} |(x, T^* y)| \leqq \|T^* y\| < +\infty.$$

Thus we obtain by Theorem 9.12

$$\gamma = \sup_{\|x\| \leqq 1} \|Tx\| < +\infty,$$

and hence $\|Tx\| \leqq \gamma \|x\|$ for every $x \in R$. Therefore we can state

Theorem 25.5. A dilatator T by $\mathfrak{R}$ is bounded if and only if the domain of T coincides with the whole space R.

Recalling the formulas (10) and (11), we see by this Theorem 25.5 that for two bounded dilatators T_1 and T_2 by $\mathfrak{R}$ we have

$$T_1 + T_2 = T_1 \oplus T_2, \qquad T_1 T_2 = T_1 \circ T_2.$$

Theorem 25.6. If a dilatator S by $\mathfrak{R}$ is bounded, then

$$T + S = T \oplus S, \qquad TS = T \circ S$$

for every dilatator T by $\mathfrak{R}$.

Proof. For any dilatator T by $\mathfrak{R}$ we can find by Theorem 24.4 a sequence $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} 1$ such that $P_\nu R$ is included in the domain of T for every $\nu = 1, 2, \ldots$. Since $TS \subset T \circ S$ by (11), if $(T \circ S)a$ has any sense, then we have for every $\nu = 1, 2, \ldots$

$$TP_\nu Sa = TSP_\nu a = (T \circ S)P_\nu a = P_\nu (T \circ S)a,$$

and hence $TSa = (T \circ S)a$, because T is closed. Therefore we have $TS = T \circ S$. We also can prove likewise by (10) that

$$T + S = T \oplus S$$

Every projection operator $P \in \mathfrak{R}$ is obviously by definition a bounded dilatator by $\mathfrak{R}$. Thus we have by Theorem 25.6

(46) $T P = T \circ P$ for every $P \in \mathfrak{R}$.

Recalling the formula (31), we obtain at once

(47) $[P] = P$ for every $P \in \mathfrak{R}$,

and hence by (37)

(48) $[T P] = [T] P$ for every $P \in \mathfrak{R}$.

Since we have by the formulas (2) and (6) in §19 for every $x \in R$

$$\left(\frac{P x}{x}, \mathfrak{F}\right) = 1 \quad \text{for } \mathfrak{F} \in \mathfrak{V}_P \mathfrak{V}_{C_{[x]}},$$

we obtain by Theorem 24.1

(49) $$(P, \mathfrak{F}) = \begin{cases} 1 & \text{for } \mathfrak{F} \in \mathfrak{V}_P \\ 0 & \text{for } \mathfrak{F} \in \mathfrak{V}_{1-P}. \end{cases}$$

Thus we have by (39) for every $\mathfrak{F} \in \mathfrak{V}_{[T]P}$

$$((T P)^{-1}, \mathfrak{F}) = \frac{1}{(T, \mathfrak{F})(P, \mathfrak{F})} = \frac{(P, \mathfrak{F})}{(T, \mathfrak{F})} = (T^{-1} P, \mathfrak{F}).$$

Furthermore we have by (41) and (48)

$$[(T P)^{-1}] = [T P] = [T] P = [T^{-1}] P = [T^{-1} P].$$

Therefore we have

(50) $(T P)^{-1} = T^{-1} P$ for every $P \in \mathfrak{R}$.

Theorem 25.7. *For two dilatators* T *and* S *by* $\mathfrak{R}$, *if*

$$T a = S a$$

then we have $T C_{[a]} = S C_{[a]}$.

Proof. If $T a = S a$, then, putting $T_1 = T \oplus (-S)$, we we have $T_1 a = 0$ by (10), and hence by (48) and (35)

$$[T_1 C_{[a]}] = [T_1] C_{[a]} = C_{[T_1][a]} = C_{[T_1 a]} = 0.$$

Thus we obtain $T_1 \circ C_{[a]} = 0$ by (46). This relation yields

$$T \circ C_{[a]} = S \circ C_{[a]}$$

by (7), and hence $T C_{[a]} = S C_{[a]}$.

Theorem 25.8. *For any pair* $a, b \in R$, *there exists uniquely a dilatator* T *by* $\mathfrak{R}$ *such that*

$$[a] b = T a, \qquad [T] = C_{[[a] b]}.$$

Proof. Putting

$$\varphi(\mathfrak{x}) = \begin{cases} \left(\frac{\mathfrak{x}}{a}, \mathfrak{x}\right) & \text{for } \mathfrak{x} \in \mathfrak{V}_{C_{[a]}} \\ 0 & \text{for } \mathfrak{x} \in \mathfrak{V}_{1-C_{[a]}}, \end{cases}$$

we obtain a dilatator by $\mathfrak{R}$

$$T = \int \varphi(\mathfrak{x}) d\mathfrak{x}.$$

For this dilatator T we have by Theorem 18.4 and the formula (36)

$$[a]\mathfrak{k} = Ta, \quad [T] \leqq C_{[a]}.$$

On the other hand we have by (35)

$$[T]\, C_{[a]} = C_{[Ta]} = C_{[[a]\mathfrak{k}]}.$$

Thus we obtain $[T] = C_{[[a]\mathfrak{k}]}$. The uniqueness of such a dilatator T is concluded immediately by Theorem 25.7.

§26 D-convergence

Let A be an open dense set in the proper space $\mathfrak{C}_{\mathfrak{R}}$ of $\mathfrak{R}$, and $\varphi_\nu(\mathfrak{x})$ $(\nu = 1, 2, \ldots)$ a sequence of finite continuous functions on A. We suppose furthermore that $\varphi_\nu(\mathfrak{x})$ $(\nu = 1, 2, \ldots)$ is convergent for every point $\mathfrak{x} \in A$. Then for any $\varepsilon > 0$, putting

$$A_\nu = \{\mathfrak{x} : |\varphi_\rho(\mathfrak{x}) - \varphi_\sigma(\mathfrak{x})| \leqq \varepsilon \quad \text{for } \rho, \sigma \geqq \nu\} \qquad (\nu = 1, 2, \ldots),$$

it is obvious that $\sum_{\nu=1}^{\infty} A_\nu = A$. Furthermore we have

$$(\sum_{\nu=1}^{\infty} A_\nu^\circ)^- = \mathfrak{C}_{\mathfrak{R}}.$$

Because, if $(\sum_{\nu=1}^{\infty} A_\nu^\circ)^- \neq \mathfrak{C}_{\mathfrak{R}}$, then we can find $0 \neq P \in \mathfrak{R}$ such that

$$\mathfrak{V}_P \subset A\,(\sum_{\nu=1}^{\infty} A_\nu^\circ)^{-\prime}.$$

For such P, we have

$$(\sum_{\nu=1}^{\infty} (A_\nu \mathfrak{V}_P)^\circ)^- = (\sum_{\nu=1}^{\infty} A_\nu^\circ)^- \mathfrak{V}_P = 0,$$

and hence $(A_\nu \mathfrak{V}_P)^\circ = 0$ for every $\nu = 1, 2, \ldots$. Since $A_\nu \mathfrak{V}_P$ is closed, $A_\nu \mathfrak{V}_P$ is nowhere dense for every $\nu = 1, 2, \ldots$, and hence $\sum_{\nu=1}^{\infty} A_\nu \mathfrak{V}_P$ also is nowhere dense by Theorem 15.10, contradicting

$$\sum_{\nu=1}^{\infty} A_\nu \mathfrak{V}_P = A\,\mathfrak{V}_P = \mathfrak{V}_P \neq 0.$$

Thus $\sum_{\nu=1}^{\infty} A_\nu^\circ$ is an open dense set in $\mathfrak{C}_{\mathfrak{R}}$.

Therefore, if we put for $\nu, \mu = 1, 2, \ldots$

$$A_{\nu,\mu} = \{\mathfrak{x} : |\varphi_\rho(\mathfrak{x}) - \varphi_\sigma(\mathfrak{x})| \leqq \frac{1}{\mu} \quad \text{for } \rho, \sigma \geqq \nu\}$$

then $\sum_{\nu=1}^{\infty} A_{\nu,\mu}^{\circ}$ is an open dense set for every $\mu = 1, 2, \ldots$, and hence, putting

$$A_0 = \prod_{\mu=1}^{\infty} \left(\sum_{\nu=1}^{\infty} A_{\nu,\mu}^{\circ} \right),$$

we obtain an open dense set A_0° by Theorem 15.11. For such A_0°, the limit

$$\varphi(\mathfrak{x}) = \lim_{\nu \to \infty} \varphi_\nu(\mathfrak{x}) \qquad (\mathfrak{x} \in A)$$

is finite continuous in A_0°. In fact, for any $\mathfrak{x}_0 \in A_0^{\circ}$ and $\varepsilon > 0$, we can find ν, μ such that

$$\mathfrak{x}_0 \in A_{\nu,\mu}^{\circ}, \qquad \frac{1}{\mu} < \frac{1}{3}\varepsilon.$$

For such ν, since φ_ν is continuous at $\mathfrak{x}_0$ by assumption, we can find an open set $B \ni \mathfrak{x}_0$ such that

$$|\varphi_\nu(\mathfrak{x}) - \varphi_\nu(\mathfrak{x}_0)| \leqq \frac{1}{3}\varepsilon \qquad \text{for } \mathfrak{x} \in B.$$

Then for every $\mathfrak{x} \in A_{\nu,\mu}^{\circ} B$ we have

$$|\varphi_\nu(\mathfrak{x}) - \varphi(\mathfrak{x})| \leqq \frac{1}{\mu}.$$

Since $\mathfrak{x}_0 \in A_{\nu,\mu}^{\circ} B$, we obtain hence naturally

$$|\varphi_\nu(\mathfrak{x}_0) - \varphi(\mathfrak{x}_0)| \leqq \frac{1}{\mu}$$

Consequently we have

$$|\varphi(\mathfrak{x}) - \varphi(\mathfrak{x}_0)| < \varepsilon \qquad \text{for every } \mathfrak{x} \in A_{\nu,\mu}^{\circ} B.$$

Therefore φ is continuous in A_0°.

Now we can hence state

Theorem 26.1. *If a sequence of finite continuous functions* φ_ν $(\nu = 1, 2, \ldots)$ *on an open dense set* A *is convergent in* A, *then the limit* $\lim_{\nu \to \infty} \varphi_\nu(\mathfrak{x})$ *also is finite continuous in some open dense set in the proper space* $\mathfrak{C}_{\mathfrak{R}}$.

For a sequence of dilatators T_ν $(\nu = 1, 2, \ldots)$ by $\mathfrak{R}$, if

$$(T_\nu, \mathfrak{x}) \qquad (\nu = 1, 2, \ldots)$$

is convergent in an open dense set A, then, putting

$$\varphi(\mathfrak{x}) = \lim_{\nu \to \infty} (T_\nu, \mathfrak{x}) \qquad \text{for } \mathfrak{x} \in A,$$

we see by Theorem 26.1 that φ is finite continuous in some open dense set, and hence, putting

$$T_0 = \int \varphi(\mathfrak{x})\, d\mathfrak{x},$$

we obtain a dilatator T_0 by $\mathfrak{R}$, for which

$$(T_0, \mathfrak{x}) = \lim_{\nu \to \infty} (T_\nu, \mathfrak{x})$$

in some open dense set. In this case, we shall say that T_ν ($\nu = 1, 2, \ldots$) is D-convergent to a limit T_0, and write

$$\text{D-}\lim_{\nu \to \infty} T_\nu = T_0 .$$

Theorem 26.2. D-convergence is independent from the ring $\mathfrak{R}$: If $\text{D-}\lim_{\nu \to \infty} T_\nu = T_0$ by $\mathfrak{R}$ and all T_ν ($\nu = 1, 2, \ldots$) are dilatators by another closed ring of projection operators $\mathfrak{S}$, then T_0 also is a dilatator by $\mathfrak{S}$ and $\text{D-}\lim_{\nu \to \infty} T_\nu = T_0$ by $\mathfrak{S}$.

Proof. We suppose firstly $\mathfrak{S} \subset \mathfrak{R}$. We have then by assumption

$$\lim_{\nu \to \infty} (T_\nu, \mathfrak{x})_{\mathfrak{R}} = (T_0, \mathfrak{x})_{\mathfrak{R}}$$

in an open dense set A in $\mathfrak{L}_{\mathfrak{R}}$. Since we have by Theorem 24.6

$$(T_\nu, \mathfrak{S}\mathfrak{x})_{\mathfrak{S}} = (T_\nu, \mathfrak{x})_{\mathfrak{R}} \qquad \text{for every } \mathfrak{x} \in \mathfrak{L}_{\mathfrak{R}},$$

$(T_\nu, \mathfrak{S}\mathfrak{x})_{\mathfrak{S}}$ ($\nu = 1, 2, \ldots$) is convergent for every $\mathfrak{x} \in A$. On the other hand, we see by Theorem 20.6 that the image of A by the partition correspondence also is an open dense set in the proper space $\mathfrak{L}_{\mathfrak{S}}$ of $\mathfrak{S}$. Thus there exists a dilatator S by $\mathfrak{S}$, for which

$$\lim_{\nu \to \infty} (T_\nu, \mathfrak{S}\mathfrak{x})_{\mathfrak{S}} = (S, \mathfrak{S}\mathfrak{x})_{\mathfrak{S}} \qquad \text{for every } \mathfrak{x} \in A .$$

Then we have $\text{D-}\lim_{\nu \to \infty} T_\nu = S$ by $\mathfrak{S}$, and $S = T_0$, because

$$(S, \mathfrak{x})_{\mathfrak{R}} = (T_0, \mathfrak{x})_{\mathfrak{R}} \qquad \text{for every } \mathfrak{x} \in A .$$

We also can prove likewise the case where $\mathfrak{R} \subset \mathfrak{S}$. In general, denoting by $\mathfrak{R}_0$ the closed ring generated by all spectral rings of T_ν ($\nu = 1, 2, \ldots$), we have obviously $\mathfrak{R}_0 \subset \mathfrak{R}\mathfrak{S}$. Thus

$$\text{D-}\lim_{\nu \to \infty} T_\nu = T_0$$

by $\mathfrak{R}$ implies the same relation by $\mathfrak{R}_0$, and hence further the same relation by $\mathfrak{S}$, as proved just above.

Concerning D-convergence, we have obviously by definition

Theorem 26.3. $\text{D-}\lim_{\nu \to \infty} T_\nu = T$, $\text{D-}\lim_{\nu \to \infty} S_\nu = S$ implies

$$\text{D-}\lim_{\nu \to \infty} (T_\nu \bullet S_\nu) = T \bullet S ,$$

$$\text{D-}\lim_{\nu \to \infty} (T_\nu \circ S_\nu) = T \circ S .$$

Theorem 26.4. For a sequence of positive dilatators

$$S_1 \geqq S_2 \geqq \cdots\cdots,$$

we have $\mathrm{D}\text{-}\lim_{\nu\to\infty} S_\nu = 0$ if and only if for the domain D_1 of S_1 we have $\lim_{\nu\to\infty} S_\nu x = 0$ for every $x \in D_1$.

Proof. By virtue of Theorem 25.3, the domain of S_ν includes D_1 for every $\nu = 2, 3, \cdots$. If $\mathrm{D}\text{-}\lim_{\nu\to\infty} S_\nu = 0$, then for each $x \in D_1$, we have by Theorem 24.1

$$\lim_{\nu\to\infty} \left(\frac{S_\nu x}{x}, \mathfrak{F}\right) = \lim_{\nu\to\infty} (S_\nu, \mathfrak{F}) = 0$$

in some open set being dense in $\mathcal{U}_{\mathcal{C}[x]}$. Thus we can find by Theorem 21.8 a sequence $\mathfrak{R} \ni P_\mu \uparrow_{\mu=1}^{\infty} C_{[x]}$ such that

$$\lim_{\nu\to\infty} P_\mu[x] S_\nu x = 0 \qquad \text{for every } \mu = 1, 2, \cdots.$$

On the other hand, recalling Theorem 25.3, we have

$$\| S_\nu x \| = \| C_{[x]} S_\nu x \| \leqq \| P_\mu S_\nu x \| + \| S_\nu (C_{[x]} - P_\mu) x \|$$
$$\leqq \| P_\mu[x] S_\nu x \| + \| S_1 (C_{[x]} - P_\mu) x \|.$$

Thus we obtain

$$\overline{\lim_{\nu\to\infty}} \| S_\nu x \| \leqq \| (C_{[x]} - P_\mu) S_1 x \|$$

for every $\mu = 1, 2, \cdots$. Making μ tend to ∞, we conclude hence

$$\lim_{\nu\to\infty} S_\nu x = 0.$$

Since $(S_1, \mathfrak{F}) \geqq (S_2, \mathfrak{F}) \geqq \cdots$ for every $\mathfrak{F} \in \mathcal{F}_{\mathfrak{R}}$ by assumption, there exists a positive dilatator S_0 by $\mathfrak{R}$ for which

$$\mathrm{D}\text{-}\lim_{\nu\to\infty} S_\nu = S_0,$$

and we have obviously $S_\nu \geqq S_0$ for every $\nu = 1, 2, \cdots$. If $S_0 \neq 0$, then there exists $a \in R$ for which $S_0 a \neq 0$. For such $a \in R$ we can find by Theorem 22.2 a sequence $\mathfrak{R} \ni P_\mu \uparrow_{\mu=1}^{\infty} C_{[a]}$ such that $P_\mu a \in D_1$ for every $\mu = 1, 2, \cdots$. Since

$$\lim_{\mu\to\infty} S_0 P_\mu a = C_{[a]} S_0 a = S_0 a \neq 0,$$

we can find μ for which $S_0 P_\mu a \neq 0$. For such μ we have by Theorem 25.3

$$\| S_\nu P_\mu a \| \geqq \| S_0 P_\mu a \| \neq 0 \qquad \text{for every } \nu = 1, 2, \cdots.$$

Therefore $\lim_{\nu\to\infty} S_\nu x = 0$ for $x \in D_1$ implies $\mathrm{D}\text{-}\lim_{\nu\to\infty} S_\nu = 0$.

Theorem 26.5. Let D_ν be the domains of dilatators T_ν ($\nu = 1, 2, \cdots$) by $\mathfrak{R}$ respectively. For any $a \in R$ we can find a sequence $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} C_{[a]}$ such that

$$P_\nu a \in \prod_{\mu=1}^{\infty} D_\mu \qquad \text{for every } \nu = 1, 2, \cdots.$$

Proof. By virtue of Theorem 22.2, for any $a \in R$ we can find a sequence $\mathfrak{R} \ni P_{\nu,\mu} \uparrow_{\nu=1}^{\infty} C_{[a]}$ $(\mu = 1, 2, \ldots)$ such that

$$P_{\nu,\mu} a \in D_\mu \qquad \text{for every } \nu, \ \mu = 1, 2, \ldots .$$

For such $P_{\nu,\mu} \uparrow_{\nu=1}^{\infty} C_{[a]}$ $(\mu = 1, 2, \ldots)$ we can find by Theorem 14.4 $\nu_{\mu,\rho}$ $(\mu, \rho = 1, 2, \ldots)$ such that

$$\bigwedge_{\mu=1}^{\infty} P_{\nu_{\mu,\rho},\mu} \uparrow_{\rho=1}^{\infty} C_{[a]}.$$

Then, putting $P_\rho = \bigwedge_{\mu=1}^{\infty} P_{\nu_{\mu,\rho},\mu}$ $(\rho = 1, 2, \ldots)$, we have

$$\mathfrak{R} \ni P_\rho \uparrow_{\rho=1}^{\infty} C_{[a]},$$

$$P_\rho a = P_\rho P_{\nu_{\mu,\rho},\mu} a \in D_\mu \qquad \text{for every } \rho, \ \mu = 1, 2, \ldots .$$

Consequently $P_\rho a \in \prod_{\mu=1}^{\infty} D_\mu$ for every $\rho = 1, 2, \ldots$.

For a sequence of positive dilatators S_ν $(\nu = 1, 2, \ldots)$ we shall write

$$S_\nu \downarrow_{\nu=1}^{\infty} 0,$$

if $S_1 \geqq S_2 \geqq \ldots$ and $D\text{-}\lim_{\nu\to\infty} S_\nu = 0$.

Theorem 26.6. *For a sequence of dilatators* T_ν $(\nu = 1, 2, \ldots)$ *by* $\mathfrak{R}$, $D\text{-}\lim_{\nu\to\infty} T_\nu = T$ *is equivalent to each one of the following conditions:*

1) *there exists a sequence of positive dilatators*

$$S_\nu \downarrow_{\nu=1}^{\infty} 0$$

such that $|T_\nu - T| \leqq S_\nu$ *for every* $\nu = 1, 2, \ldots$,

2) *for each* $a \in R$ *we can find a sequence*

$$\mathfrak{R} \ni P_\mu \uparrow_{\mu=1}^{\infty} C_{[a]}$$

such that $(\mathfrak{R})\text{-}\lim_{\nu\to\infty} T_\nu P_\mu a = T P_\mu a$ *for every* $\mu = 1, 2, \ldots$,

3) *there exists a system* $a_\lambda \in R$ $(\lambda \in \Lambda)$ *such that*

$$\bigcup_{\lambda\in\Lambda} C_{[a_\lambda]} = 1, \quad (\mathfrak{R})\text{-}\lim_{\nu\to\infty} T_\nu a_\lambda = T a_\lambda \text{ for every } \lambda \in \Lambda,$$

4) *we can find a sequence*

$$\mathfrak{R} \ni P_\lambda \uparrow_{\lambda\in\Lambda} 1$$

such that $\lim_{\nu\to\infty} \| T_\nu P_\lambda - T P_\lambda \| = 0$ *for every* $\lambda \in \Lambda$.

Proof. By virtue of Theorem 26.3, we need only consider the

case where $T = 0$. If $D\text{-}\lim_{\nu\to\infty} T_\nu = 0$, then we can find by definition an open dense set A in the proper space $\mathfrak{L}_\mathfrak{R}$ such that

$$\lim_{\nu\to\infty} (T_\nu, \mathfrak{x}) = 0 \qquad \text{for every } \mathfrak{x} \in A.$$

Thus, putting

$$\varphi_\nu(\mathfrak{x}) = \sup_{\mu \geqq \nu} |(T_\mu, \mathfrak{x})| \qquad \text{for } \mathfrak{x} \in A,$$

we have obviously for every $\mathfrak{x} \in A$

$$\varphi_1(\mathfrak{x}) \geqq \varphi_2(\mathfrak{x}) \geqq \cdots, \qquad \lim_{\nu\to\infty} \varphi_\nu(\mathfrak{x}) = 0.$$

On the other hand we have obviously

$$\varphi_\nu(\mathfrak{x}) = \lim_{\rho\to\infty} \{ \operatorname{Max}_{\nu \leqq \mu \leqq \rho} |(T_\mu, \mathfrak{x})| \} \qquad \text{for every } \mathfrak{x} \in A.$$

Thus we see by Theorem 26.1 that φ_ν is finite continuous in some open dense set, and hence for each $\nu = 1, 2, \ldots$, we can find a positive dilatator S_ν by $\mathfrak{R}$ such that $(S_\nu, \mathfrak{x}) = \varphi_\nu(\mathfrak{x})$ in some open dense set. For such S_ν $(\nu = 1, 2, \ldots)$, we see easily by Theorem 15.11 that

$$S_\nu \downarrow_{\nu=1}^{\infty} 0, \qquad |T_\nu| \leqq S_\nu \qquad (\nu = 1, 2, \ldots).$$

Therefore $D\text{-}\lim_{\nu\to\infty} T_\nu = T$ implies 1).

Next we suppose that $S_\nu \downarrow_{\nu=1}^{\infty} 0$ and $|T_\nu| \leqq S_\nu$ for every $\nu = 1, 2, \ldots$. For any $a \in R$ we can find by Theorem 26.5 a sequence $\mathfrak{R} \ni P_\mu \uparrow_{\mu=1}^{\infty} C_{[a]}$ such that all $T_\nu P_\mu a$ and $S_\nu P_\mu a$ $(\nu, \mu = 1, 2, \ldots)$ have sense. Then we have by Theorem 26.4

$$\lim_{\nu\to\infty} S_\nu P_\mu a = 0 \qquad \text{for every } \mu = 1, 2, \ldots.$$

On the other hand we have by Theorem 25.3 for every $P \in \mathfrak{R}$

$$\| T_\nu P P_\mu a \| \leqq \| S_\nu P P_\mu a \|, \quad \| S_{\nu+1} P P_\mu a \| \leqq \| S_\nu P P_\mu a \|,$$

and hence for every $\mu = 1, 2, \ldots$

$$T_\nu P_\mu a \prec S_\nu P_\mu a \qquad (\nu = 1, 2, \ldots),$$

$$S_1 P_\mu a \succ S_2 P_\mu a \succ \cdots\cdots.$$

Thus we have $(\mathfrak{R})\text{-}\lim_{\nu\to\infty} T_\nu P_\mu a = 0$ by Theorem 21.7 for every $\mu = 1, 2, \ldots$. Therefore 1) implies 2). Since $\bigcup_{x\in R} C_{[x]} = 1$ by Theorem 14.1, 2) implies obviously 3).

Now we suppose that there exists a system $a_\lambda \in R$ $(\lambda \in \Lambda)$ such that $\bigcup_{\lambda\in\Lambda} C_{[a_\lambda]} = 1$ and

$$(\mathfrak{R})\text{-}\lim_{\nu\to\infty} T_\nu a_\lambda = 0 \qquad \text{for every } \lambda \in \Lambda.$$

Then we can find by definition a system $l_\lambda \in R$ $(\lambda \in \Lambda)$ such that for any $\varepsilon > 0$ and $\lambda \in \Lambda$ there exists ν_0 for which

$$\| T_\nu P a_\lambda \| \leqq \varepsilon \| P l_\lambda \| \quad \text{for } \nu \geqq \nu_0, \ P \in \mathfrak{R}.$$

For such $f_\lambda \in R$ $(\lambda \in \Lambda)$, putting

$$\varphi(\mathfrak{x}) = \lim_{P \to \mathfrak{x}} \frac{\|P f_\lambda\|^2}{\|P a_\lambda\|^2} \qquad \text{for } \mathfrak{x} \in \mathcal{U}_{\mathcal{C}_{[a_\lambda]}},$$

we see by Theorem 18.2 that φ is a continuous function on $\mathcal{C}_{[a_\lambda]}$ and finite in some open set being dense in $\mathcal{C}_{[a_\lambda]}$. Furthermore we have by the formula §19(12)

$$|(T_\nu, \mathfrak{x})| \leqq \varepsilon \sqrt{\varphi(\mathfrak{x})} \qquad \text{for } \mathfrak{x} \in \mathcal{C}_{[a_\lambda]},\ \nu \geqq \nu_0.$$

Thus, putting

$$\mathcal{U}_{P_\mu} = \{\mathfrak{x} : \varphi(\mathfrak{x}) < \mu\}^- \qquad (\mu = 1, 2, \ldots),$$

we have $\mathfrak{R} \ni P_\mu \uparrow_{\mu=1}^{\infty} \mathcal{C}_{[a_\lambda]}$ and

$$\lim_{\nu \to \infty} \|T_\nu P_\mu\| = 0 \qquad \text{for every } \mu = 1, 2, \ldots.$$

On the other hand, if

$$\lim_{\nu \to \infty} \|T_\nu P\| = \lim_{\nu \to \infty} \|T_\nu Q\| = 0$$

for $P, Q \in \mathfrak{R}$, then we also have

$$\lim_{\nu \to \infty} \|T_\nu (P \vee Q)\| = 0,$$

because $P \vee Q = P + (Q - PQ)$ and

$$\|T_\nu (P \vee Q)\| \leqq \|T_\nu P\| + \|T_\nu Q\| \|1 - P\| \leqq \|T_\nu P\| + \|T_\nu Q\|.$$

Since $\bigcup_{\lambda \in \Lambda} \mathcal{C}_{[a_\lambda]} = 1$, we conclude hence that there exists a system

$$\mathfrak{R} \ni P_\lambda \uparrow_{\lambda \in \Lambda} 1$$

for which $\lim_{\nu \to \infty} \|T_\nu P_\lambda\| = 0$ for every $\lambda \in \Lambda$. Therefore 3) implies 4).

Finally we suppose that there exists a system $\mathfrak{R} \ni P_\lambda \uparrow_{\lambda \in \Lambda} 1$ for which

$$\lim_{\nu \to \infty} \|T_\nu P_\lambda\| = 0 \qquad \text{for every } \lambda \in \Lambda.$$

Since $|(T_\nu, \mathfrak{x})| \leqq \|T_\nu P_\lambda\|$ for $\mathfrak{x} \in \mathcal{U}_{P_\lambda}$ by Theorem 25.4 and the formula §25(49), we have

$$\lim_{\nu \to \infty} |(T_\nu, \mathfrak{x})| = 0 \qquad \text{for } \mathfrak{x} \in \sum_{\lambda \in \Lambda} \mathcal{U}_{P_\lambda}.$$

Since $\sum_{\lambda \in \Lambda} \mathcal{U}_{P_\lambda}$ is an open dense set by Theorem 15.7, we conclude hence by definition $\text{D-}\lim_{\nu \to \infty} T_\nu = 0$.

By virtue of 1) in Theorem 26.6, we see that if a sequence of dilatators T_ν $(\nu = 1, 2, \ldots)$ by $\mathfrak{R}$ is D-convergent, then there exists a positive dilatator S by $\mathfrak{R}$ for which

$$|T_\nu| \leqq S \qquad \text{for every } \nu = 1, 2, \ldots.$$

Theorem 26.7. *For a positive dilatator S by $\mathfrak{R}$ with domain D,*

$$|T_\nu| \leqq S \quad (\nu = 1, 2, \ldots); \qquad \text{D-}\lim_{\nu \to \infty} T_\nu = T$$

implies

$$|T| \leqq S\,, \qquad (\mathfrak{R})\text{-}\lim_{\nu\to\infty} T_\nu x = Tx \quad \text{for every } x \in D.$$

Proof. We have by assumption

$$|(T_\nu, \mathfrak{x})| \leqq (S, \mathfrak{x}) \quad \text{for every } \mathfrak{x} \in \mathfrak{C}_{\mathfrak{R}},$$

$$\lim_{\nu\to\infty} (T_\nu, \mathfrak{x}) = (T, \mathfrak{x}) \quad \text{in some open dense set.}$$

Thus we also have

$$|(T, \mathfrak{x})| \leqq (S, \mathfrak{x}) \quad \text{for every } \mathfrak{x} \in \mathfrak{C}_{\mathfrak{R}},$$

because both sides are continuous over the whole space $\mathfrak{C}_{\mathfrak{R}}$. Therefore we have $|T| \leqq S$ by definition.

Since $D\text{-}\lim_{\nu\to\infty} T_\nu = T$ by assumption, we can find by Theorem 26.6 a sequence of positive dilatators $S_\nu \downarrow_{\nu=1}^{\infty} 0$ such that

$$|T_\nu - T| \leqq S_\nu \quad \text{for every } \nu = 1, 2, \ldots .$$

Then for each $\nu = 1, 2, \ldots$ we have by definition

$$|(T_\nu, \mathfrak{x}) - (T, \mathfrak{x})| \leqq (S_\nu, \mathfrak{x})$$

in some open dense set. On the other hand we have

$$|(T_\nu, \mathfrak{x}) - (T, \mathfrak{x})| \leqq 2(S, \mathfrak{x})$$

in some open dense set. Thus, putting

$$\varphi_\nu(\mathfrak{x}) = \operatorname{Min}\{(S_\nu, \mathfrak{x}), 2(S, \mathfrak{x})\} \qquad (\nu = 1, 2, \ldots),$$

we obtain a sequence of continuous functions φ_ν $(\nu = 1, 2, \ldots)$ on $\mathfrak{C}_{\mathfrak{R}}$ such that

$$|(T_\nu, \mathfrak{x}) - (T, \mathfrak{x})| \leqq \varphi_\nu(\mathfrak{x}) \qquad (\nu = 1, 2, \ldots)$$

in some open dense set. Since

$$2(S, \mathfrak{x}) \geqq \varphi_1(\mathfrak{x}) \geqq \varphi_2(\mathfrak{x}) \geqq \cdots \quad \text{for every } \mathfrak{x} \in \mathfrak{C}_{\mathfrak{R}},$$

putting

$$K_\nu = \int \varphi_\nu(\mathfrak{x})\, d\mathfrak{x} \qquad (\nu = 1, 2, \ldots),$$

we have $2S \geqq K_1 \geqq K_2 \geqq \cdots$ and for each $x \in D$ we conclude easily

$$T_\nu x - Tx \prec K_\nu x \quad \text{for every } \nu = 1, 2, \ldots .$$

Furthermore we have obviously

$$K_\nu \leqq S_\nu \quad \text{for every } \nu = 1, 2, \ldots,$$

and hence we also have $K_\nu \downarrow_{\nu=1}^{\infty} 0$. Thus we obtain by Theorem 26.4

$$\lim_{\nu\to\infty} K_\nu x = 0 \quad \text{for every } x \in D.$$

Therefore we conclude by Theorem 21.7

$$(\mathfrak{R})\text{-}\lim_{\nu\to\infty} T_\nu x = Tx \quad \text{for every } x \in D.$$

As a special case of Theorem 25.7, we have by Theorem 25.4

Theorem 26.8. If $\mathrm{D}\text{-}\lim_{\nu\to\infty} T_\nu = T$ and $\|T_\nu\| \leqq \gamma$ for every $\nu = 1, 2, \ldots$, then we also have $\|T\| \leqq \gamma$ and

$$(\mathfrak{R})\text{-}\lim_{\nu\to\infty} T_\nu x = Tx \quad \text{for every } x \in R.$$

§27 Measurable Functions

Let $\mathfrak{M}$ be a totally additive set class in an abstract space Ω, that is,

1) $\Omega \in \mathfrak{M}$,

2) $\mathfrak{E} \in \mathfrak{M}$ implies $\mathfrak{E}' \in \mathfrak{M}$,

3) $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ implies $\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu \in \mathfrak{M}$.

From 2) and 3) we conclude at once

4) $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ implies $\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu \in \mathfrak{M}$.

A finite function $\varphi(\zeta)$ $(\zeta \in \Omega)$ on Ω is said to be $(\mathfrak{M})$-measurable, if

$$\{\zeta : \varphi(\zeta) \in \Lambda^0\} \in \mathfrak{M}$$

for every open set Λ^0 in the complex plane G.

Theorem 27.1. A finite function φ on Ω is $(\mathfrak{M})$-measurable, if and only if for any $\varepsilon > 0$ we can find a sequence of sets $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ such that

$$\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu = \Omega, \qquad \sup_{\xi,\eta \in \mathfrak{E}_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \quad (\nu = 1, 2, \ldots).$$

Proof. We set

$$\Lambda_{\alpha,\varepsilon} = \{\zeta : |\zeta - \alpha| < \varepsilon\}$$

for a complex number α and a positive number ε. If φ is $(\mathfrak{M})$-measurable, then for any $\varepsilon > 0$ we can find a sequence of complex numbers α_ν $(\nu = 1, 2, \ldots)$ such that

$$G = \sum_{\nu=1}^{\infty} \Lambda_{\alpha_\nu, \frac{1}{2}\varepsilon},$$

and, putting

$$\mathfrak{E}_\nu = \{\zeta : \varphi(\zeta) \in \Lambda_{\alpha_\nu, \frac{1}{2}\varepsilon}\} \qquad (\nu = 1, 2, \ldots),$$

we have

$$\Omega = \sum_{\nu=1}^{\infty} \Xi_\nu, \qquad \sup_{\xi,\eta \in \Xi_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \quad (\nu = 1, 2, \ldots).$$

Conversely we suppose that for any $\varepsilon > 0$ we can find a sequence $\Xi_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ such that

$$\Omega = \sum_{\nu=1}^{\infty} \Xi_\nu, \qquad \sup_{\xi,\eta \in \Xi_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \quad (\nu = 1, 2, \ldots).$$

For an arbitrary complex number α and $0 < \delta' < \delta$, we can find a sequence $\Xi_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ such that

$$\Omega = \sum_{\nu=1}^{\infty} \Xi_\nu, \qquad \sup_{\xi,\eta \in \Xi_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \frac{1}{2}(\delta - \delta').$$

Then, putting

$$\Psi_{\delta'} = \sum_{\{\nu : |\varphi(\zeta) - \alpha| < \delta'\} \Xi_\nu \neq 0} \Xi_\nu,$$

we see easily that $\Psi_{\delta'} \in \mathfrak{M}$ and

$$\{\zeta : |\varphi(\zeta) - \alpha| < \delta'\} \subset \Psi_{\delta'} \subset \{\zeta : |\varphi(\zeta) - \alpha| < \delta\}.$$

Since

$$\{\zeta : |\varphi(\zeta) - \alpha| < \delta\} = \sum_{\nu=1}^{\infty} \{\zeta : |\varphi(\zeta) - \alpha| < (1 - \tfrac{1}{\nu})\delta\},$$

we have

$$\{\zeta : |\varphi(\zeta) - \alpha| < \delta\} = \sum_{\nu=1}^{\infty} \Psi_{(1-\frac{1}{\nu})\delta} \in \mathfrak{M}.$$

For any open set $\Lambda^\circ \neq 0$ in the complex plane G, we can find complex numbers α_ν and $\delta_\nu > 0$ $(\nu = 1, 2, \ldots)$ such that

$$\Lambda^\circ = \sum_{\nu=1}^{\infty} \Lambda_{\alpha_\nu, \delta_\nu}$$

and then we have

$$\{\zeta : \varphi(\zeta) \in \Lambda^\circ\} = \sum_{\nu=1}^{\infty} \{\zeta : |\varphi(\zeta) - \alpha_\nu| < \delta_\nu\} \in \mathfrak{M}.$$

Therefore φ is $(\mathfrak{M})$-measurable.

Theorem 27.2. For a sequence of $(\mathfrak{M})$-measurable functions φ_ν $(\nu = 1, 2, \ldots)$ on Ω, if $\lim_{\nu \to \infty} \varphi_\nu(\zeta) = \varphi(\zeta)$ for every $\zeta \in \Omega$, then φ also is $(\mathfrak{M})$-measurable.

Proof. For a complex number α and a positive number ε, putting

$$\Xi_{\alpha,\varepsilon} = \sum_{\mu=1}^{\infty} \prod_{\nu \geqq \mu} \{\zeta : |\varphi_\nu(\zeta) - \alpha| < \varepsilon\},$$

we see easily that $\Xi_{\alpha,\varepsilon} \in \mathfrak{M}$ and

$$\{\zeta : |\varphi(\zeta) - \alpha| < \varepsilon\} \subset \Xi_{\alpha,\varepsilon} \subset \{\zeta : |\varphi(\zeta) - \alpha| \leqq \varepsilon\}.$$

For an arbitrary open set $\Lambda^\circ \neq 0$ in the complex plane G, we can find

complex numbers α_ν and positive numbers ε_ν $(\nu = 1, 2, \ldots)$ such that

$$\Lambda^\circ = \sum_{\nu=1}^{\infty} \{ \xi : |\xi - \alpha_\nu| < \varepsilon_\nu \} = \sum_{\nu=1}^{\infty} \{ \xi : |\xi - \alpha_\nu| \leqq \varepsilon_\nu \},$$

and then we have

$$\{ \zeta : \varphi(\zeta) \in \Lambda^\circ \} = \sum_{\nu=1}^{\infty} \mathfrak{E}_{\alpha_\nu, \varepsilon_\nu} \in \mathfrak{M}.$$

Therefore φ is $(\mathfrak{M})$-measurable by definition.

For a point set $\mathfrak{E}$ in Ω, the function

$$\chi(\zeta) = \begin{cases} 1 & \text{for } \zeta \in \mathfrak{E} \\ 0 & \text{for } \zeta \bar{\in} \mathfrak{E} \end{cases}$$

is called the characteristic function of $\mathfrak{E}$ and denoted by $\chi_{\mathfrak{E}}$.

For every $\mathfrak{E} \in \mathfrak{M}$, the characteristic function $\chi_{\mathfrak{E}}$ of $\mathfrak{E}$ is $(\mathfrak{M})$-measurable by definition. If a function φ on Ω is $(\mathfrak{M})$-measurable, then for every $\mathfrak{E} \in \mathfrak{M}$, the product $\varphi \chi_{\mathfrak{E}}$ also is $(\mathfrak{M})$-measurable. Because, for any $\varepsilon > 0$ we can find by Theorem 27.1 a sequence $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ such that

$$\Omega = \sum_{\nu=1}^{\infty} \mathfrak{E}_\nu, \qquad \sup_{\xi, \eta \in \mathfrak{E}_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \quad (\nu = 1, 2, \ldots),$$

and then we have $\Omega = \sum_{\nu=1}^{\infty} \mathfrak{E}_\nu \mathfrak{E} + \sum_{\nu=1}^{\infty} \mathfrak{E}_\nu \mathfrak{E}'$,

$$\sup_{\xi, \eta \in \mathfrak{E}_\nu \mathfrak{E}} |\varphi(\xi)\chi_{\mathfrak{E}}(\xi) - \varphi(\eta)\chi_{\mathfrak{E}}(\eta)| \leqq \sup_{\xi, \eta \in \mathfrak{E}_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon,$$

$$\sup_{\xi, \eta \in \mathfrak{E}_\nu \mathfrak{E}'} |\varphi(\xi)\chi_{\mathfrak{E}}(\xi) - \varphi(\eta)\chi_{\mathfrak{E}}(\eta)| = 0$$

for every $\nu = 1, 2, \ldots$. Thus $\varphi \chi_{\mathfrak{E}}$ is $(\mathfrak{M})$-measurable by Theorem 27.1.

If both φ and ψ are $(\mathfrak{M})$-measurable functions on Ω, then for any $\varepsilon > 0$ we can find by Theorem 27.1 two sequences $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ and $\mathfrak{F}_\mu \in \mathfrak{M}$ $(\mu = 1, 2, \ldots)$ such that

$$\Omega = \sum_{\nu=1}^{\infty} \mathfrak{E}_\nu = \sum_{\nu=1}^{\infty} \mathfrak{F}_\nu,$$

$$\sup_{\xi, \eta \in \mathfrak{E}_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \frac{1}{2}\varepsilon,$$

$$\sup_{\xi, \eta \in \mathfrak{F}_\mu} |\psi(\xi) - \psi(\eta)| \leqq \frac{1}{2}\varepsilon$$

for every $\nu, \mu = 1, 2, \ldots$, and hence

$$\sup_{\xi, \eta \in \mathfrak{E}_\nu \mathfrak{F}_\mu} |(\varphi(\xi) + \psi(\xi)) - (\varphi(\eta) + \psi(\eta))| \leqq \varepsilon \qquad (\nu, \mu = 1, 2, \ldots).$$

Since $\Omega = \sum_{\nu,\mu=1}^{\infty} \mathfrak{E}_\nu \mathfrak{F}_\mu$, we conclude therefore by Theorem 27.1 that

$\varphi + \psi$ also is $(\mathfrak{M})$-measurable.

If two $(\mathfrak{M})$-measurable functions φ and ψ are bounded and

$$|\varphi(\xi)| \leqq \gamma, \qquad |\psi(\xi)| \leqq \gamma \quad \text{for every } \xi \in \Omega,$$

then for any $\varepsilon > 0$ we can find $\Phi_\nu, \Psi_\mu \in \mathfrak{M}$ $(\nu, \mu = 1, 2, \ldots)$ such that

$$\Omega = \sum_{\nu=1}^{\infty} \Phi_\nu = \sum_{\mu=1}^{\infty} \Psi_\mu,$$

$$\sup_{\xi, \eta \in \Phi_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \frac{1}{2\gamma}\varepsilon \qquad (\nu = 1, 2, \ldots),$$

$$\sup_{\xi, \eta \in \Psi_\mu} |\psi(\xi) - \psi(\eta)| \leqq \frac{1}{2\gamma}\varepsilon \qquad (\mu = 1, 2, \ldots),$$

and then we have

$$\sup_{\xi, \eta \in \Phi_\nu \Psi_\mu} |\varphi(\xi)\psi(\xi) - \varphi(\eta)\psi(\eta)|$$
$$\leqq \sup_{\xi, \eta \in \Phi_\nu \Psi_\mu} |\varphi(\xi)|\,|\psi(\xi) - \psi(\eta)| + \sup_{\xi, \eta \in \Phi_\nu \Psi_\mu} |\psi(\eta)|\,|\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon.$$

Since $\Omega = \sum_{\nu,\mu=1}^{\infty} \Phi_\nu \Psi_\mu$, we conclude hence by Theorem 27.1 that $\varphi\psi$ also is $(\mathfrak{M})$-measurable.

For two $(\mathfrak{M})$-measurable functions φ and ψ, putting

$$\Phi_\nu = \{\xi : |\varphi(\xi)| < \nu\}, \qquad \Psi_\nu = \{\xi : |\psi(\xi)| < \nu\},$$

we have obviously $\Phi_\nu, \Psi_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$,

$$|\varphi(\xi)\chi_{\Phi_\nu}(\xi)| \leqq \nu, \quad |\psi(\xi)\chi_{\Psi_\nu}(\xi)| \leqq \nu \quad \text{for every } \xi \in \Omega,$$

and

$$\lim_{\nu \to \infty} (\varphi(\xi)\chi_{\Phi_\nu}(\xi))\,(\psi(\xi)\chi_{\Psi_\nu}(\xi)) = \varphi(\xi)\psi(\xi).$$

Since both $\varphi\chi_{\Phi_\nu}$ and $\psi\chi_{\Psi_\nu}$ are bounded $(\mathfrak{M})$-measurable functions, $(\varphi\chi_{\Phi_\nu})(\psi\chi_{\Psi_\nu})$ also is $(\mathfrak{M})$-measurable, as proved just above. Thus we conclude by Theorem 27.2 that $\varphi\psi$ also is $(\mathfrak{M})$-measurable.

Therefore we have

Theorem 27.3. *If both φ and ψ are $(\mathfrak{M})$-measurable functions on Ω, then $\varphi + \psi$ and $\varphi\psi$ also are $(\mathfrak{M})$-measurable.*

For a finite function φ on Ω, we define $\frac{1}{\varphi}$ to mean

$$\frac{1}{\varphi}(\xi) = \begin{cases} \dfrac{1}{\varphi(\xi)} & \text{for } \varphi(\xi) \neq 0 \\ 0 & \text{for } \varphi(\xi) = 0. \end{cases}$$

With this definition, we have obviously

$$\varphi \frac{1}{\varphi} = \chi_{\{\xi :\, \varphi(\xi) \neq 0\}}.$$

Theorem 27.4. If a functior φ on Ω is $(\mathfrak{M})$-measurable, then $\frac{1}{\varphi}$ also is $(\mathfrak{M})$-measurable.

Proof. Putting $\mathfrak{E}_\nu = \{\zeta : |\varphi(\zeta)| > \frac{1}{\nu}\}$, we have obviously $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ and

$$\lim_{\nu \to \infty} \frac{1}{\varphi \chi_{\mathfrak{E}_\nu}}(\zeta) = \frac{1}{\varphi}(\zeta) \qquad \text{for every } \zeta \in \Omega .$$

For any $\varepsilon > 0$ we can find by Theorem 27.1 a sequence $\mathfrak{T}_\mu \in \mathfrak{M}$ $(\mu = 1, 2, \ldots)$ such that $\Omega = \sum_{\mu=1}^{\infty} \mathfrak{T}_\mu$ and

$$\sup_{\xi, \eta \in \mathfrak{T}_\mu} |\varphi(\xi)\chi_{\mathfrak{E}_\nu}(\xi) - \varphi(\eta)\chi_{\mathfrak{E}_\nu}(\eta)| \leqq \frac{1}{\nu^2}\varepsilon$$

for every $\mu = 1, 2, \ldots$. Then we have

$$\sup_{\xi, \eta \in \mathfrak{T}_\mu \mathfrak{E}_\nu} \left| \frac{1}{\varphi \chi_{\mathfrak{E}_\nu}}(\xi) - \frac{1}{\varphi \chi_{\mathfrak{E}_\nu}}(\eta) \right| \leqq \varepsilon,$$

$$\sup_{\xi, \eta \in \mathfrak{T}_\mu \mathfrak{E}_\nu'} \left| \frac{1}{\varphi \chi_{\mathfrak{E}_\nu}}(\xi) - \frac{1}{\varphi \chi_{\mathfrak{E}_\nu}}(\eta) \right| = 0 .$$

Since

$$\Omega = \sum_{\mu=1}^{\infty} \mathfrak{T}_\mu \mathfrak{E}_\nu + \sum_{\mu=1}^{\infty} \mathfrak{T}_\mu \mathfrak{E}_\nu' ,$$

we see by Theorem 27.1 that $\frac{1}{\varphi \chi_{\mathfrak{E}_\nu}}$ is $(\mathfrak{M})$-measurable for every $\nu = 1, 2, \ldots$. Therefore $\frac{1}{\varphi}$ also is $(\mathfrak{M})$-measurable by Theorem 27.2.

Theorem 27.5. If a function φ on Ω is $(\mathfrak{M})$-measurable, then for any Borel set Λ in the complex plane G we have

$$\{\zeta : \varphi(\zeta) \in \Lambda\} \in \mathfrak{M} .$$

Proof. Denoting by $\mathfrak{K}$ the class of those point sets Λ in G for which

$$\{\zeta : \varphi(\zeta) \in \Lambda\} \in \mathfrak{M},$$

we see easily that $\mathfrak{K}$ is totally additive and contains all open sets in G . Thus $\mathfrak{K}$ includes all Borel sets in G .

A finite function ω on the complex plane G is said to be Borel measurable, if the point set

$$\{\zeta : \omega(\zeta) \in \Lambda^\circ\}$$

is a Borel set for every open set Λ° in G .

With this definition, we have obviously by Theorem 27.5

Theorem 27.6. If a function φ on Ω is $(\mathfrak{M})$-measurable, then

for any Borel measurable function ω on the complex plane G,

$$\omega(\varphi(\zeta)) \qquad (\zeta \in \Omega)$$

also is a $(\mathfrak{M})$-measurable function on Ω.

§28 General Spectralities

Let $\mathfrak{M}$ be a totally additive set class in an abstract space Ω. A system of projection operators $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ is called a spectrality, if

1) $E(\mathfrak{E})E(\mathfrak{F}) = E(\mathfrak{E}\mathfrak{F})$,

2) $E(\mathfrak{E}) + E(\mathfrak{E}') = 1$,

3) $E(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu) = \bigcup_{\nu=1}^{\infty} E(\mathfrak{E}_\nu)$.

From 1) we conclude immediately

(4) $E(\mathfrak{E})E(\mathfrak{F}) = E(\mathfrak{F})E(\mathfrak{E})$,

(5) $\mathfrak{E} \supset \mathfrak{F}$ implies $E(\mathfrak{E}) \geq E(\mathfrak{F})$.

Since $E(\mathfrak{E})E(\mathfrak{E}') = 0$ by 2), we obtain by 1) and 2)

(6) $E(0) = 0$, $\quad E(\Omega) = 1$,

and hence further

(7) $\mathfrak{E}\mathfrak{F} = 0$ implies $E(\mathfrak{E})E(\mathfrak{F}) = 0$.

We have obviously by (5) and 3)

(8) $\mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \dots$ implies $E(\mathfrak{E}_\nu) \uparrow_{\nu=1}^{\infty} E(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu)$.

From 2) and 3) we conclude easily

(9) $E(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu) = \bigcap_{\nu=1}^{\infty} E(\mathfrak{E}_\nu)$,

(10) $\mathfrak{E}_1 \supset \mathfrak{E}_2 \supset \dots$ implies $E(\mathfrak{E}_\nu) \downarrow_{\nu=1}^{\infty} E(\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu)$.

Let E be a spectrality on $\mathfrak{M}$ and φ a $(\mathfrak{M})$-measurable function on Ω. For a point set $\mathfrak{E}_0 \in \mathfrak{M}$, if φ is bounded in $\mathfrak{E}_0$ and

$$|\varphi(\zeta)| \leq \gamma \qquad \text{for every } \zeta \in \mathfrak{E}_0,$$

then for any $\varepsilon > 0$ we can find a finite number of complex numbers α_ν $(\nu = 1, 2, \dots, \varkappa)$ for which

$$\{\zeta : |\zeta| \leq \gamma\} \subset \sum_{\nu=1}^{\varkappa} \{\zeta : |\zeta - \alpha_\nu| < \tfrac{1}{2}\varepsilon\},$$

and putting

$$\mathfrak{E}_\nu = \{\zeta : |\varphi(\zeta) - \alpha_\nu| < \tfrac{1}{2}\varepsilon\} \quad (\nu = 1, 2, \dots, \varkappa),$$

we have obviously

$$\mathfrak{E}_\nu \in \mathfrak{M}, \quad \mathfrak{E}_0 \subset \sum_{\nu=1}^{\varkappa} \mathfrak{E}_\nu \quad (\nu = 1, 2, \dots, \varkappa).$$

$$\sup_{\xi,\eta \in \mathfrak{E}_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon$$

For such $\mathfrak{E}_\nu$ $(\nu = 1, 2, \dots, \varkappa)$, putting

$$\mathfrak{F}_1 = \mathfrak{E}_0\mathfrak{E}_1, \qquad \mathfrak{F}_\nu = \mathfrak{E}_0\mathfrak{E}_\nu\Big(\sum_{\mu=1}^{\nu-1}\mathfrak{E}_\mu\Big)' \quad (\nu = 2, 3, \dots, \varkappa),$$

we obtain a partition of $\mathfrak{E}_0$:

$$\mathfrak{E}_0 = \sum_{\nu=1}^{\varkappa} \mathfrak{F}_\nu, \qquad \mathfrak{F}_\nu\mathfrak{F}_\mu = 0 \quad \text{for } \nu \neq \mu,$$

such that $\mathfrak{F}_\nu \in \mathfrak{M}$ and

$$\sup_{\xi,\eta \in \mathfrak{F}_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \qquad (\nu = 1, 2, \dots, \varkappa).$$

For two partitions of $\mathfrak{E}_0$ in $\mathfrak{M}$:

$$\mathfrak{E}_0 = \sum_{\nu=1}^{\varkappa} \mathfrak{E}_\nu, \quad \sup_{\xi,\eta \in \mathfrak{E}_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \quad (\nu = 1, 2, \dots, \varkappa),$$

$$\mathfrak{E}_0 = \sum_{\mu=1}^{\rho} \mathfrak{F}_\mu, \quad \sup_{\xi,\eta \in \mathfrak{F}_\mu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \quad (\mu = 1, 2, \dots, \rho)$$

and for $\xi_\nu \in \mathfrak{E}_\nu$, $\eta_\mu \in \mathfrak{F}_\mu$, we have for every $a \in R$

$$\begin{aligned} &\Big\| \sum_{\nu=1}^{\varkappa} \varphi(\xi_\nu) E(\mathfrak{E}_\nu) a - \sum_{\mu=1}^{\rho} \varphi(\eta_\mu) E(\mathfrak{F}_\mu) a \Big\|^2 \\ &= \sum_{\nu,\mu=1}^{\varkappa,\rho} |\varphi(\xi_\nu) - \varphi(\eta_\mu)|^2 \, \| E(\mathfrak{E}_\nu\mathfrak{F}_\mu) a \|^2 \\ &\leqq 4\varepsilon^2 \sum_{\nu,\mu=1}^{\varkappa,\rho} \| E(\mathfrak{E}_\nu\mathfrak{F}_\mu) a \|^2 = 4\varepsilon^2 \| E(\mathfrak{E}_0) a \|^2, \end{aligned}$$

because $\mathfrak{E}_\nu = \sum_{\mu=1}^{\rho} \mathfrak{E}_\nu\mathfrak{F}_\mu$, $\mathfrak{F}_\mu = \sum_{\nu=1}^{\varkappa} \mathfrak{E}_\nu\mathfrak{F}_\mu$, and

$$|\varphi(\xi_\nu) - \varphi(\eta_\mu)| \leqq 2\varepsilon \qquad \text{for } \mathfrak{E}_\nu\mathfrak{F}_\mu \neq 0.$$

Therefore there exists the limit

$$\lim_{\varepsilon \to 0} \sum_{\nu=1}^{\varkappa} \varphi(\zeta_\nu) E(\mathfrak{E}_\nu) a$$

for $\mathfrak{E}_0 = \sum_{\nu=1}^{\varkappa} \mathfrak{E}_\nu$, $\mathfrak{E}_\nu\mathfrak{E}_\mu = 0$ for $\nu \neq \mu$, $\zeta_\nu \in \mathfrak{E}_\nu$, and

$$\sup_{\xi,\eta \in \mathfrak{E}_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \qquad (\nu = 1, 2, \dots, \varkappa)$$

This limit is called the integral of φ by $E(\mathfrak{E})a$ $(\mathfrak{E} \in \mathfrak{M})$ in $\mathfrak{E}_0$ and denoted by

$$\int_{\mathfrak{E}_0} \varphi(\zeta) E(d\zeta) a.$$

With this definition we have obviously for every $\mathfrak{E} \in \mathfrak{M}$

$$E(\mathfrak{E}) \int_{\mathfrak{E}_0} \varphi(\zeta) E(d\zeta) a = \int_{\mathfrak{E}\mathfrak{E}_0} \varphi(\zeta) E(d\zeta) a. \tag{11}$$

Thus, if $\Xi_1 \subset \Xi_2 \subset \cdots$, $\Xi_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$, and $\Xi_0 = \sum_{\nu=1}^{\infty} \Xi_\nu$, then we have by (8)

$$\lim_{\nu\to\infty} \int_{\Xi_\nu} \varphi(\zeta) E(d\zeta) a = \lim_{\nu\to\infty} E(\Xi_\nu) \int_{\Xi_0} \varphi(\zeta) E(d\zeta) a$$
$$= E(\Xi_0) \int_{\Xi_0} \varphi(\zeta) E(d\zeta) a = \int_{\Xi_0} \varphi(\zeta) E(d\zeta) a.$$

In the case where φ is not bounded in Ξ_0, putting

$$\Xi_\nu = \{\zeta : |\varphi(\zeta)| < \nu\} \qquad (\nu = 1, 2, \ldots),$$

we obtain a sequence $\Xi_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ such that

$$\Xi_1 \subset \Xi_2 \subset \cdots, \qquad \Xi_0 = \sum_{\nu=1}^{\infty} \Xi_0 \Xi_\nu,$$
$$\sup_{\zeta \in \Xi_0 \Xi_\nu} |\varphi(\zeta)| \leqq \nu \qquad (\nu = 1, 2, \ldots).$$

If there exists a sequence $\Xi_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ such that

$$\Xi_1 \subset \Xi_2 \subset \cdots, \qquad \Xi_0 = \sum_{\nu=1}^{\infty} \Xi_\nu,$$

φ is bounded in Ξ_ν for every $\nu = 1, 2, \ldots$, and

$$\int_{\Xi_\nu} \varphi(\zeta) E(d\zeta) a \qquad (\nu = 1, 2, \ldots)$$

is convergent, then for every sequence $\Psi_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ subject to the condition: $\Psi_1 \subset \Psi_2 \subset \cdots$, $\Xi_0 = \sum_{\nu=1}^{\infty} \Psi_\nu$, and φ is bounded in Ψ_ν for every $\nu = 1, 2, \ldots$, the sequence

$$\int_{\Psi_\nu} \varphi(\zeta) E(d\zeta) a \qquad (\nu = 1, 2, \ldots)$$

is convergent always to the same limit. In fact, we have by (11) for every $\mu = 1, 2, \ldots$

$$E(\Xi_\nu) \int_{\Psi_\mu} \varphi(\zeta) E(d\zeta) a = E(\Psi_\mu) \int_{\Xi_\nu} \varphi(\zeta) E(d\zeta) a,$$

and hence

$$\int_{\Psi_\mu} \varphi(\zeta) E(d\zeta) a = E(\Xi_0) \int_{\Psi_\mu} \varphi(\zeta) E(d\zeta) a$$
$$= \lim_{\nu\to\infty} E(\Xi_\nu) \int_{\Psi_\mu} \varphi(\zeta) E(d\zeta) a = E(\Psi_\mu) \lim_{\nu\to\infty} \int_{\Xi_\nu} \varphi(\zeta) E(d\zeta) a.$$

Thus, making μ tend to ∞, we obtain

$$\lim_{\mu\to\infty} \int_{\Psi_\mu} \varphi(\zeta) E(d\zeta) a = E(\Xi_0) \lim_{\nu\to\infty} \int_{\Xi_\nu} \varphi(\zeta) E(d\zeta) a$$
$$= \lim_{\nu\to\infty} \int_{\Xi_\nu} \varphi(\zeta) E(d\zeta) a.$$

Therefore, in this case we shall say that φ is integrable by $E(\mathfrak{E})a$ $(\mathfrak{E} \in \mathfrak{M})$ in $\mathfrak{E}_0$, and the same limit will be denoted by

$$\int_{\mathfrak{E}_0} \varphi(\zeta) E(d\zeta)\, a .$$

If a $(\mathfrak{M})$-measurable function φ is bounded in $\mathfrak{E}_0 \in \mathfrak{M}$, then we have by definition for every $a, b \in R$

$$\lim_{\varepsilon \to 0} \sum_{\nu=1}^{\varkappa} \varphi(\zeta_\nu)\, (E(\mathfrak{E}_\nu)a, b) = \left(\int_{\mathfrak{E}_0} \varphi(\zeta) E(d\zeta)\, a, b\right)$$

for partitions of $\mathfrak{E}_0$ in $\mathfrak{M}$:

$$\mathfrak{E}_0 = \sum_{\nu=1}^{\varkappa} \mathfrak{E}_\nu, \quad \mathfrak{E}_\nu \mathfrak{E}_\mu = 0 \text{ for } \nu \neq \mu, \quad \zeta_\nu \in \mathfrak{E}_\nu,$$

$$\sup_{\xi, \eta \in \mathfrak{E}_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \qquad (\nu = 1, 2, \ldots, \varkappa).$$

Thus we define the integral of φ by $(E(d\zeta)a, b)$ in $\mathfrak{E}_0$ as

$$\int_{\mathfrak{E}_0} \varphi(\zeta)\, (E(d\zeta)a, b) = \lim_{\varepsilon \to 0} \sum_{\nu=1}^{\varkappa} \varphi(\zeta_\nu)\, (E(\mathfrak{E}_\nu)a, b)$$

for partitions of $\mathfrak{E}_0$ in $\mathfrak{M}$:

$$\mathfrak{E}_0 = \sum_{\nu=1}^{\varkappa} \mathfrak{E}_\nu, \quad \mathfrak{E}_\nu \mathfrak{E}_\mu = 0 \text{ for } \nu \neq \mu, \quad \zeta_\nu \in \mathfrak{E}_\nu,$$

$$\sup_{\xi, \eta \in \mathfrak{E}_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \qquad (\nu = 1, 2, \ldots, \varkappa).$$

With this definition we have obviously

$$\int_{\mathfrak{E}_0} \varphi(\zeta)\, (E(d\zeta)a, b) = \left(\int_{\mathfrak{E}_0} \varphi(\zeta) E(d\zeta)\, a, b\right), \tag{12}$$

and further

$$\int_{\mathfrak{E}_0} \varphi(\zeta) \overline{\psi(\zeta)}\, (E(d\zeta)a, b) = \left(\int_{\mathfrak{E}_0} \varphi(\zeta) E(d\zeta)\, a, \int_{\mathfrak{E}_0} \psi(\zeta) E(d\zeta)\, b\right). \tag{13}$$

Indeed, we have by definition

$$\begin{aligned}
&\left(\int_{\mathfrak{E}_0} \varphi(\zeta) E(d\zeta)\, a, \int_{\mathfrak{E}_0} \psi(\zeta) E(d\zeta)\, b\right) \\
&\quad = \lim_{\varepsilon \to 0} \left(\sum_{\nu=1}^{\varkappa} \varphi(\zeta_\nu) E(\mathfrak{E}_\nu)a, \sum_{\nu=1}^{\varkappa} \psi(\zeta_\nu) E(\mathfrak{E}_\nu)b\right) \\
&\quad = \lim_{\varepsilon \to 0} \sum_{\nu=1}^{\varkappa} \varphi(\zeta_\nu) \overline{\psi(\zeta_\nu)}\, (E(\mathfrak{E}_\nu)a, b) \\
&\quad = \int_{\mathfrak{E}_0} \varphi(\zeta) \overline{\psi(\zeta)}\, (E(d\zeta)a, b),
\end{aligned}$$

because $E(\mathfrak{E}_\nu)E(\mathfrak{E}_\mu) = 0$ for $\nu \neq \mu$.

From (13) we conclude immediately

$$(14)\qquad \Big\| \int_{\mathfrak{F}_0} \varphi(\zeta) E(d\zeta) a \Big\|^2 = \int_{\mathfrak{F}_0} |\varphi(\zeta)|^2 \, \| E(d\zeta) a \|^2.$$

For a $(\mathfrak{M})$-measurable function φ, if for every sequence $\mathfrak{F}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ such that $\mathfrak{F}_0 = \sum_{\nu=1}^{\infty} \mathfrak{F}_\nu$, $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \ldots$, and φ is bounded in $\mathfrak{F}_\nu$ for every $\nu = 1, 2, \ldots$, the sequence

$$\int_{\mathfrak{F}_\nu} \varphi(\zeta) \, (E(d\zeta) a, \ell) \qquad (\nu = 1, 2, \ldots)$$

is convergent to the same limit, then we shall say that φ is integrable by $(E(\mathfrak{F})a, \ell)$ $(\mathfrak{F} \in \mathfrak{M})$ in $\mathfrak{F}_0$ and this limit will be denoted by

$$\int_{\mathfrak{F}_0} \varphi(\zeta) \, (E(d\zeta) a, \ell).$$

With this definition we have obviously by (13)

Theorem 28.1. If a function φ is integrable by $E(\mathfrak{F})a$ $(\mathfrak{F} \in \mathfrak{M})$ in $\mathfrak{F}_0$ and a function ψ is integrable by $E(\mathfrak{F})\ell$ $(\mathfrak{F} \in \mathfrak{M})$ in $\mathfrak{F}_0$, then $\varphi(\zeta)\overline{\psi(\zeta)}$ $(\zeta \in \mathfrak{F}_0)$ is integrable by $(E(\mathfrak{F})a, \ell)$ $(\mathfrak{F} \in \mathfrak{M})$ in $\mathfrak{F}_0$ and

$$\int_{\mathfrak{F}_0} \varphi(\zeta)\overline{\psi(\zeta)} \, (E(d\zeta) a, \ell) = \Big(\int_{\mathfrak{F}_0} \varphi(\zeta) E(d\zeta) a, \int_{\mathfrak{F}_0} \psi(\zeta) E(d\zeta) \ell \Big).$$

Theorem 28.2. A $(\mathfrak{M})$-measurable function φ is integrable by $E(\mathfrak{F})a$ $(\mathfrak{F} \in \mathfrak{M})$ in $\mathfrak{F}_0$ if and only if $|\varphi(\zeta)|^2$ $(\zeta \in \mathfrak{F}_0)$ is integrable by $\| E(\mathfrak{F}) a \|^2$ $(\mathfrak{F} \in \mathfrak{M})$ in $\mathfrak{F}_0$ and then

$$\Big\| \int_{\mathfrak{F}_0} \varphi(\zeta) E(d\zeta) a \Big\|^2 = \int_{\mathfrak{F}_0} |\varphi(\zeta)|^2 \, \| E(d\zeta) a \|^2.$$

Proof. If φ is integrable by $E(\mathfrak{F})a$ $(\mathfrak{F} \in \mathfrak{M})$ in $\mathfrak{F}_0$, then we see by Theorem 28.1 that $|\varphi(\zeta)|^2$ is integrable by $\| E(\mathfrak{F})a \|^2$ $(\mathfrak{F} \in \mathfrak{M})$ in $\mathfrak{F}_0$ and

$$\Big\| \int_{\mathfrak{F}_0} \varphi(\zeta) E(d\zeta) a \Big\|^2 = \int_{\mathfrak{F}_0} |\varphi(\zeta)|^2 \, \| E(d\zeta) a \|^2.$$

Conversely, if there is a sequence $\mathfrak{F}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ such that $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \ldots$, $\mathfrak{F}_0 = \sum_{\nu=1}^{\infty} \mathfrak{F}_\nu$, φ is bounded in $\mathfrak{F}_\nu$ for every $\nu = 1, 2, \ldots$, and

$$\sup_{\nu \geq 1} \int_{\mathfrak{F}_\nu} |\varphi(\zeta)|^2 \, \| E(d\zeta) a \|^2 < +\infty,$$

then, putting $\Psi_0 = 0$, we have by (14) and (11)

$$\int_{\Psi_\nu} |\varphi(\xi)|^2 \| E(d\xi)a \|^2 = \| \int_{\Psi_\nu} \varphi(\xi) E(d\xi)a \|^2$$
$$= \| \sum_{\mu=1}^{\nu} \{ E(\Psi_\mu) - E(\Psi_{\mu-1}) \} \int_{\Psi_\nu} \varphi(\xi) E(d\xi)a \|^2$$
$$= \sum_{\mu=1}^{\nu} \| E(\Psi_\mu - \Psi_{\mu-1}) \int_{\Psi_\nu} \varphi(\xi) E(d\xi)a \|^2$$
$$= \sum_{\mu=1}^{\nu} \| \int_{\Psi_\mu - \Psi_{\mu-1}} \varphi(\xi) E(d\xi)a \|^2,$$

because $E(\Psi_\mu - \Psi_{\mu-1}) E(\Psi_\rho - \Psi_{\rho-1}) = 0$ for $\mu \neq \rho$. Recalling Theorem 7.11, we can find hence by assumption $\beta \in R$ such that

$$\beta = \sum_{\nu=1}^{\infty} \int_{\Psi_\mu - \Psi_{\mu-1}} \varphi(\xi) E(d\xi)a = \lim_{\nu \to \infty} \int_{\Psi_\nu} \varphi(\xi) E(d\xi)a.$$

Therefore φ is integrable by $E(\mathfrak{E})a$ ($\mathfrak{E} \in \mathfrak{M}$) in $\mathfrak{E}_0$ by definition.

§29 Integral Operators

Let $E(\mathfrak{E})$ ($\mathfrak{E} \in \mathfrak{M}$) be a spectrality on a totally additive set set class in an abstract space Ω. We suppose furthermore that for a closed ring of projection operators $\mathcal{R} \ni 1$ we have

$$E(\mathfrak{E}) \in \mathcal{R} \qquad \text{for every } \mathfrak{E} \in \mathfrak{M}.$$

For a $(\mathfrak{M})$-measurable function φ on Ω, putting

$$\Psi_\nu = \{ \xi : |\varphi(\xi)| < \nu \} \qquad (\nu = 1, 2, \ldots),$$

we have obviously $\Psi_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$. For a point $\mathfrak{x}$ of the proper space $\mathfrak{X}_{\mathcal{R}}$ of $\mathcal{R}$, if $\mathfrak{x} \in \mathfrak{V}_{E(\Psi_\nu)}$, then we can find a complex number α such that for any $\varepsilon > 0$ there exists $\mathfrak{E} \in \mathfrak{M}$ for which

$$\mathfrak{x} \in \mathfrak{V}_{E(\mathfrak{E})}, \qquad |\varphi(\xi) - \alpha| \leqq \varepsilon \qquad \text{for every } \xi \in \mathfrak{E}.$$

Because for any $\rho = 1, 2, \ldots$ we can find obviously a finite number of complex numbers α_μ $(\mu = 1, 2, \ldots, \varkappa)$ such that

$$\{ \xi : |\varphi(\xi)| < \nu \} \subset \sum_{\mu=1}^{\varkappa} \{ \xi : |\varphi(\xi) - \alpha_\mu| < \frac{1}{2^{\rho+1}} \},$$

and then we have by Theorem 15.3

$$E(\{ \xi : |\varphi(\xi) - \alpha_\mu| < \frac{1}{2^{\rho+1}} \}) \in \mathfrak{x} \qquad \text{for some } \mu.$$

For such μ, putting

$$\mathfrak{E}_\rho = \{ \xi : |\varphi(\xi) - \alpha_\mu| < \frac{1}{2^{\rho+1}} \} \qquad (\rho = 1, 2, \ldots),$$

we obtain a sequence $\mathfrak{E}_\rho \in \mathfrak{M}$ $(\rho = 1, 2, \ldots)$ such that

$$\mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{F}_\rho)}, \qquad \sup_{\xi,\eta \in \mathfrak{F}_\rho} |\varphi(\xi) - \varphi(\eta)| \leqq \frac{1}{2^\rho}.$$

For such $\mathfrak{F}_\rho \in \mathfrak{M}$ ($\rho = 1, 2, \ldots$), since $\mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{F}_\rho)}$ for every $\rho = 1, 2, \ldots$, we have $\mathfrak{F}_\rho \mathfrak{F}_\sigma \neq 0$ for every $\rho, \sigma = 1, 2, \ldots$, and hence for any $\zeta_\rho \in \mathfrak{F}_\rho$ ($\rho = 1, 2, \ldots$) we have

$$|\varphi(\zeta_\rho) - \varphi(\zeta_{\rho+1})| \leqq \frac{1}{2^\rho} + \frac{1}{2^{\rho+1}} = \frac{3}{2^{\rho+1}}$$

for every $\rho = 1, 2, \ldots$. Thus $\varphi(\zeta_\rho)$ ($\rho = 1, 2, \ldots$) is convergent, and putting

$$\alpha = \lim_{\rho \to \infty} \varphi(\zeta_\rho),$$

we have

$$|\varphi(\zeta) - \alpha| \leqq \frac{1}{2^\rho} \qquad \text{for } \zeta \in \mathfrak{F}_\rho,$$

because for every $\sigma = 1, 2, \ldots$ we have

$$|\varphi(\zeta) - \varphi(\zeta_\sigma)| \leqq \frac{1}{2^\rho} + \frac{1}{2^\sigma} \quad \text{for every } \zeta \in \mathfrak{F}_\rho$$

and $\lim_{\sigma \to \infty} \varphi(\zeta_\sigma) = \alpha$. Therefore for any $\varepsilon > 0$ we can find $\mathfrak{F} \in \mathfrak{M}$ such that

$$\mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{F})}, \qquad |\varphi(\zeta) - \alpha| \leqq \varepsilon \qquad \text{for every } \zeta \in \mathfrak{F}.$$

The uniqueness of such α is evident. This complex number α will be denoted by

$$\lim_{E\mathfrak{M} \to \mathfrak{F}} \varphi.$$

For any $\mathfrak{F} \in \sum_{\nu=1}^{\infty} \mathfrak{U}_{E(\mathfrak{F}_\nu)}$ there exists hence $\lim_{E\mathfrak{M} \to \mathfrak{F}} \varphi$. For $\mathfrak{F} \in \left(\sum_{\nu=1}^{\infty} \mathfrak{U}_{E(\mathfrak{F}_\nu)}\right)'$, we define

$$\lim_{E\mathfrak{M} \to \mathfrak{F}} \varphi = \infty,$$

because for every $\nu = 1, 2, \ldots$ we have $\mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{F}_\nu')}$ and

$$|\varphi(\zeta)| \geqq \nu \qquad \text{for every } \zeta \in \mathfrak{F}_\nu'.$$

Now, putting

$$\omega_\varphi(\mathfrak{F}) = \lim_{E\mathfrak{M} \to \mathfrak{F}} \varphi \qquad \text{for every } \mathfrak{F} \in \mathfrak{C}_\mathcal{R},$$

we obtain a continuous function ω_φ on $\mathfrak{C}_\mathcal{R}$. In fact, if $\omega_\varphi(\mathfrak{F}_0)$ is finite, then for any $\varepsilon > 0$ we can find $\mathfrak{F} \in \mathfrak{M}$ such that

$$\mathfrak{F}_0 \in \mathfrak{U}_{E(\mathfrak{F})}, \qquad |\varphi(\zeta) - \omega_\varphi(\mathfrak{F}_0)| \leqq \frac{1}{2}\varepsilon \qquad (\zeta \in \mathfrak{F}).$$

Then for any $\mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{F})}$, since $\mathfrak{F} \subset \mathfrak{F}_\nu$ for some ν, we can find $\mathfrak{F}$ $\in \mathfrak{M}$ such that

$$\mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{F})}, \qquad |\varphi(\zeta) - \omega_\varphi(\mathfrak{F})| \leqq \frac{1}{2}\varepsilon \qquad (\zeta \in \mathfrak{F}),$$

and hence

$$|\omega_\varphi(\mathfrak{F}) - \omega_\varphi(\mathfrak{F}_0)| \leqq \varepsilon,$$

because $E(\mathfrak{E})E(\mathfrak{F}) \neq 0$. Thus ω_φ is continuous at $\mathfrak{F}_0$.

If $\omega_\varphi(\mathfrak{F}_0) = \infty$, then for any $\nu = 1, 2, \ldots$ we can find $\mathfrak{E} \in \mathfrak{M}$ such that

$$\mathfrak{F}_0 \in \mathfrak{U}_{E(\mathfrak{E})}, \quad |\varphi(\zeta)| \geqq 2\nu \quad \text{for every } \zeta \in \mathfrak{E}.$$

For an arbitrary point $\mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{E})}$, if $\omega_\varphi(\mathfrak{F})$ is finite, then we can find $\mathfrak{F} \in \mathfrak{M}$ such that

$$\mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{F})}, \quad |\varphi(\zeta) - \omega_\varphi(\mathfrak{F})| \leqq 1 \quad \text{for every } \zeta \in \mathfrak{F},$$

and hence

$$|\omega_\varphi(\mathfrak{F})| \geqq 2\nu - 1 \geqq \nu,$$

because $E(\mathfrak{E})E(\mathfrak{F}) \neq 0$. Therefore ω_φ is continuous at $\mathfrak{F}_0$.

Furthermore ω_φ is finite in an open dense set $\sum_{\nu=1}^{\infty} \mathfrak{U}_{E(\mathfrak{F}_\nu)}$, because $E(\mathfrak{F}_\nu) \uparrow_{\nu=1}^{\infty} E(\Omega) = 1$ by the formulas (6) and (8) in §28.

Thus we can state now

Theorem 29.1. *For a $(\mathfrak{M})$-measurable function φ on Ω, putting*

$$\omega_\varphi(\mathfrak{F}) = \lim_{E\mathfrak{M} \to \mathfrak{F}} \varphi \quad \text{for } \mathfrak{F} \in \mathfrak{S}_\mathfrak{R},$$

we obtain a continuous function ω_φ on $\mathfrak{S}_\mathfrak{R}$ such that ω_φ is finite in an open dense set; if $\omega_\varphi(\mathfrak{F})$ is finite, then for any $\varepsilon > 0$ we can find $\mathfrak{E} \in \mathfrak{M}$ such that

$$\mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{E})}, \quad |\varphi(\zeta) - \omega_\varphi(\mathfrak{F})| \leqq \varepsilon \quad \text{for every } \zeta \in \mathfrak{E};$$

and if $\omega_\varphi(\mathfrak{F}) = \infty$, then for any $\varepsilon > 0$ we can find $\mathfrak{E} \in \mathfrak{M}$ such that

$$\mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{E})}, \quad |\varphi(\zeta)| \geqq \varepsilon \quad \text{for every } \zeta \in \mathfrak{E}.$$

Furthermore we have

Theorem 29.2. *For every $\mathfrak{E} \in \mathfrak{M}$ we have*

$$\sup_{\xi,\eta \in \mathfrak{E}} |\varphi(\xi) - \varphi(\eta)| = \sup_{\zeta \in \mathfrak{E},\, \mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{E})}} |\varphi(\zeta) - \omega_\varphi(\mathfrak{F})|$$
$$= \sup_{\mathfrak{F},\mathfrak{G} \in \mathfrak{U}_{E(\mathfrak{E})}} |\omega_\varphi(\mathfrak{F}) - \omega_\varphi(\mathfrak{G})|.$$

Proof. If $|\varphi(\xi) - \varphi(\eta)| \leqq \delta$ for every $\xi, \eta \in \mathfrak{E}$, then for each $\mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{E})}$, $\omega_\varphi(\mathfrak{F})$ is obviously finite, and for any $\varepsilon > 0$ we can find $\mathfrak{F} \in \mathfrak{M}$ such that

$$\mathfrak{F} \in \mathfrak{U}_{E(\mathfrak{F})}, \quad |\varphi(\zeta) - \omega_\varphi(\mathfrak{F})| \leqq \varepsilon \quad \text{for every } \zeta \in \mathfrak{F}.$$

since $E(\mathfrak{X})E(\mathfrak{Y}) \neq 0$, we conclude hence

$$|\varphi(\xi) - \omega_\varphi(\mathfrak{f})| \leqq \delta + \varepsilon \quad \text{for every } \xi \in \mathfrak{X},\ \mathfrak{f} \in \mathfrak{U}_{E(\mathfrak{Y})}.$$

Since $\varepsilon > 0$ may be arbitrary, we obtain therefore

$$|\varphi(\xi) - \omega_\varphi(\mathfrak{f})| \leqq \delta \quad \text{for every } \xi \in \mathfrak{X},\ \mathfrak{f} \in \mathfrak{U}_{E(\mathfrak{Y})}.$$

Conversely, if

$$|\varphi(\xi) - \omega_\varphi(\mathfrak{f})| \leqq \delta \quad \text{for every } \xi \in \mathfrak{Y},\ \mathfrak{f} \in \mathfrak{U}_{E(\mathfrak{X})},$$

then for any $\eta \in \mathfrak{X}$ and $\varepsilon > 0$, putting

$$\mathfrak{Z} = \{\zeta : |\varphi(\zeta) - \varphi(\eta)| < \varepsilon\} \mathfrak{X},$$

we have obviously $\eta \in \mathfrak{Z} \in \mathfrak{M}$ and further

$$|\varphi(\eta) - \omega_\varphi(\mathfrak{f})| \leqq 2\varepsilon \quad \text{for } \mathfrak{f} \in \mathfrak{U}_{E(\mathfrak{Z})},$$

as proved just above. Since $\mathfrak{U}_{E(\mathfrak{Z})} \subset \mathfrak{U}_{E(\mathfrak{X})}$, we obtain hence

$$|\varphi(\xi) - \varphi(\eta)| \leqq \delta + 2\varepsilon \quad \text{for every } \xi, \eta \in \mathfrak{X}.$$

Since $\varepsilon > 0$ may be arbitrary, we conclude thus

$$|\varphi(\xi) - \varphi(\eta)| \leqq \delta \quad \text{for every } \xi, \eta \in \mathfrak{X}.$$

Therefore we have

$$\sup_{\xi,\eta \in \mathfrak{X}} |\varphi(\xi) - \varphi(\eta)| = \sup_{\zeta \in \mathfrak{X},\ \mathfrak{f} \in \mathfrak{U}_{E(\mathfrak{X})}} |\varphi(\zeta) - \omega_\varphi(\mathfrak{f})|.$$

If $|\varphi(\zeta) - \omega_\varphi(\mathfrak{f})| \leqq \delta$ for $\zeta \in \mathfrak{X}$, $\mathfrak{f} \in \mathfrak{U}_{E(\mathfrak{X})}$, then for any $\eta \in \mathfrak{U}_{E(\mathfrak{X})}$ and $\varepsilon > 0$ we can find $\mathfrak{Z} \in \mathfrak{M}$ for which

$$\eta \in \mathfrak{U}_{E(\mathfrak{Z})}, \quad |\varphi(\zeta) - \omega_\varphi(\eta)| \leqq \varepsilon \quad \text{for every } \zeta \in \mathfrak{Z}.$$

Since $E(\mathfrak{X})E(\mathfrak{Z}) \neq 0$, we obtain hence

$$|\omega_\varphi(\mathfrak{f}) - \omega_\varphi(\eta)| \leqq \delta + \varepsilon \quad \text{for every } \mathfrak{f}, \eta \in \mathfrak{U}_{E(\mathfrak{X})}.$$

Since $\varepsilon > 0$ may be arbitrary, we conclude therefore

$$|\omega_\varphi(\mathfrak{f}) - \omega_\varphi(\eta)| \leqq \delta \quad \text{for every } \mathfrak{f}, \eta \in \mathfrak{U}_{E(\mathfrak{X})}.$$

Conversely, if

$$|\omega_\varphi(\mathfrak{f}) - \omega_\varphi(\eta)| \leqq \delta \quad \text{for every } \mathfrak{f}, \eta \in \mathfrak{U}_{E(\mathfrak{Y})},$$

then for every $\xi \in \mathfrak{X}$ and $\varepsilon > 0$, putting

$$\mathfrak{Z} = \{\zeta : |\varphi(\zeta) - \varphi(\xi)| < \varepsilon\} \mathfrak{X},$$

we have obviously $\xi \in \mathfrak{Z} \in \mathfrak{M}$ and further

$$|\varphi(\xi) - \omega_\varphi(\mathfrak{f})| \leqq 2\varepsilon \quad \text{for every } \mathfrak{f} \in \mathfrak{U}_{E(\mathfrak{Z})},$$

as proved just above. Since $\mathfrak{U}_{E(\mathfrak{Z})} \subset \mathfrak{U}_{E(\mathfrak{X})}$, we obtain hence

$$|\varphi(\xi) - \omega_\varphi(\eta)| \leqq \delta + 2\varepsilon \quad \text{for every } \xi \in \mathfrak{X},\ \eta \in \mathfrak{U}_{E(\mathfrak{X})}.$$

Since $\varepsilon > 0$ may be arbitrary, we obtain therefore

$$|\varphi(\xi) - \omega_\varphi(\eta)| \leqq \delta \quad \text{for every } \xi \in \mathfrak{X},\ \eta \in \mathfrak{U}_{E(\mathfrak{X})}.$$

Thus we conclude

$$\sup_{\xi\in\mathfrak{X},\,\mathfrak{x}\in\mathfrak{U}_{E(\mathfrak{X})}} |\varphi(\xi)-\omega_\varphi(\mathfrak{x})| = \sup_{\mathfrak{x},\mathfrak{y}\in\mathfrak{U}_{E(\mathfrak{X})}} |\omega_\varphi(\mathfrak{x})-\omega_\varphi(\mathfrak{y})|.$$

For a $(\mathfrak{M})$-measurable function φ on Ω, denoting by D the totality of those elements $x\in R$ for which φ is integrable by $E(\mathfrak{X})x$ $(\mathfrak{X}\in\mathfrak{M})$ in Ω, we obtain an operator T with domain D as

$$Tx = \int_\Omega \varphi(\xi)E(d\xi)x \qquad \text{for } x\in D.$$

This operator T is called the integral operator of φ by a spectrality E on $\mathfrak{M}$ and denoted by

$$\int \varphi(\xi)E(d\xi).$$

Theorem 29.3. If $E(\mathfrak{X})\in\mathfrak{R}$ for every $\mathfrak{X}\in\mathfrak{M}$, then the integral operator

$$T = \int \varphi(\xi)E(d\xi)$$

is a dilatator by $\mathfrak{R}$ and we have

$$(T,\mathfrak{x}) = \lim_{E\mathfrak{M}\to\mathfrak{x}} \varphi \qquad \text{for every } \mathfrak{x}\in\mathfrak{L}_{\mathfrak{R}}.$$

Proof. If φ is bounded in some $\mathfrak{X}_0\in\mathfrak{M}$, then we have by the definition of integral for every $x\in R$

$$\int_{\mathfrak{X}_0} \varphi(\xi)E(d\xi)x = \lim_{\varepsilon\to 0} \sum_{\nu=1}^{\varkappa} \varphi(\zeta_\nu)E(\mathfrak{X}_\nu)x$$

for partitions of $\mathfrak{X}_0$ in $\mathfrak{M}$:

$$\mathfrak{X}_0 = \sum_{\nu=1}^{\varkappa} \mathfrak{X}_\nu, \quad \mathfrak{X}_\nu\mathfrak{X}_\mu = 0 \text{ for } \nu\neq\mu, \quad \zeta_\nu\in\mathfrak{X}_\nu\in\mathfrak{M},$$

$$\sup_{\xi,\eta\in\mathfrak{X}} |\varphi(\xi)-\varphi(\eta)| \leqq \varepsilon \qquad (\nu = 1, 2, \ldots, \varkappa).$$

For such partitions $\mathfrak{X}_\nu$ $(\nu = 1, 2, \ldots, \varkappa)$, we also have by Theorem 29.2

$$\sup_{\mathfrak{x},\mathfrak{y}\in\mathfrak{U}_{E(\mathfrak{X}_\nu)}} |\omega_\varphi(\mathfrak{x})-\omega_\varphi(\mathfrak{y})| \leqq \varepsilon \qquad (\nu = 1, 2, \ldots, \varkappa).$$

Thus we obtain by the definition of integral

$$\int_{E(\mathfrak{X}_0)} \omega_\varphi(\mathfrak{x})d\mathfrak{x}\,x = \lim_{\varepsilon\to 0} \sum_{\nu=1}^{\varkappa} \omega_\varphi(\mathfrak{x}_\nu)E(\mathfrak{X}_\nu)x$$

for $\mathfrak{x}_\nu\in\mathfrak{U}_{E(\mathfrak{X}_\nu)}$ $(\nu = 1, 2, \ldots, \varkappa)$. On the other hand we have by Theorem 29.2

$$|\varphi(\zeta_\nu)-\omega_\varphi(\mathfrak{x}_\nu)| \leqq \varepsilon \qquad (\nu = 1, 2, \ldots, \varkappa),$$

because $\zeta_\nu\in\mathfrak{X}_\nu$, $\mathfrak{x}_\nu\in\mathfrak{U}_{E(\mathfrak{X}_\nu)}$ $(\nu = 1, 2, \ldots, \varkappa)$. Consequently we have

$$\| \sum_{\nu=1}^{\varkappa} \varphi(\zeta_\nu) E(\mathfrak{E}_\nu) x - \sum_{\nu=1}^{\varkappa} \omega_\varphi(\xi_\nu) E(\mathfrak{E}_\nu) x \|^2$$
$$= \sum_{\nu=1}^{\varkappa} | \varphi(\zeta_\nu) - \omega_\varphi(\xi_\nu) |^2 \| E(\mathfrak{E}_\nu) x \|^2$$
$$\leq \varepsilon^2 \sum_{\nu=1}^{\varkappa} \| E(\mathfrak{E}_\nu) x \|^2 = \varepsilon^2 \| E(\mathfrak{E}_0) x \|^2.$$

Therefore we conclude

$$\int_{\mathfrak{E}_0} \varphi(\zeta) E(d\zeta) x = \int_{E(\mathfrak{E}_0)} \omega_\varphi(\xi) d\xi x.$$

For a sequence $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ subject to the condition : $\mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \ldots$, $\Omega = \sum_{\nu=1}^{\varkappa} \mathfrak{E}_\nu$, and φ is bounded in $\mathfrak{E}_\nu$ for every $\nu = 1, 2, \ldots$, we have by the formula §28(8) $E(\mathfrak{E}_\nu) \uparrow_{\nu=1}^{\infty} 1$, and by Theorem 29.2 that ω_φ is bounded in $\mathfrak{O}_{E(\mathfrak{E}_\nu)}$ for every $\nu = 1, 2, \ldots$. Furthermore we have for every $x \in R$

$$\int_{\mathfrak{E}_\nu} \varphi(\zeta) E(d\zeta) x = \int_{E(\mathfrak{E}_\nu)} \omega_\varphi(\xi) d\xi x \quad (\nu = 1, 2, \ldots).$$

Therefore we conclude by the definition of integral operators

$$\int \varphi(\zeta) E(d\zeta) = \int \omega_\varphi(\xi) d\xi,$$

and hence T is a dilatator by $\mathfrak{R}$ by Theorem 22.3. Furthermore we have by Theorem 24.1

$$(T, \xi) = \omega_\varphi(\xi) \qquad \text{for every } \xi \in \mathfrak{k}_{\mathfrak{R}},$$

because $\omega_\varphi(\xi)$ is continuous over the whole space $\mathfrak{k}_{\mathfrak{R}}$.

Theorem 29.4. For every integral operator

$$T = \int \varphi(\zeta) E(d\zeta).$$

we have

$$T^* = \int \overline{\varphi(\zeta)}\, E(d\zeta).$$

Proof. Since $(T, \xi) = \lim_{E\mathfrak{M} \to \xi} \varphi$ by Theorem 29.3, putting

$$\overline{\varphi}(\zeta) = \overline{\varphi(\zeta)} \qquad \text{for every } \zeta \in \Omega,$$

we have obviously by definition

$$\overline{(T, \xi)} = \lim_{E\mathfrak{M} \to \xi} \overline{\varphi}$$

Thus we obtain by Theorem 29.3

$$\int \overline{\varphi(\zeta)}\, E(d\zeta) = \int \overline{(T, \xi)}\, d\xi.$$

On the other hand we have by the formula §25(12)

$$T^* = \int \overline{(T, \xi)}\, d\xi,$$

and hence

$$T^* = \int \overline{\varphi(\zeta)}\, E(d\zeta).$$

Theorem 29.5. For an integral operator

$$T = \int \varphi(\zeta) E(d\zeta),$$

if we denote by E_T the spectrality of T as a dilatator, then

$$E_T(\Lambda) = E(\{\zeta : \varphi(\zeta) \in \Lambda\})$$

for every Borel set Λ in the complex plane G.

Proof. Putting $\omega_\varphi(\mathfrak{z}) = \lim_{E\mathfrak{M}\to\mathfrak{z}} \varphi$ for $\mathfrak{z} \in \mathfrak{L}_{\mathfrak{M}}$, we see easily by definition that if φ is bounded in some $\mathfrak{E} \in \mathfrak{M}$, then

$$\omega_\varphi(\mathfrak{z}) \in \{\varphi(\zeta) : \zeta \in \mathfrak{E}\}^- \qquad \text{for every } \mathfrak{z} \in \mathfrak{U}_{E(\mathfrak{E})}.$$

Thus, putting $\mathfrak{E}_\Lambda = \{\zeta : \varphi(\zeta) \in \Lambda\}$ for arbitrary point set Λ in the complex plane G, we have for every bounded closed set Λ^- in G

$$\omega_\varphi(\mathfrak{z}) \in \Lambda^- \qquad \text{for every } \mathfrak{z} \in \mathfrak{U}_{E(\mathfrak{E}_{\Lambda^-})},$$

and hence

$$\{\mathfrak{z} : \omega_\varphi(\mathfrak{z}) \in \Lambda^-\}^\circ \supset \mathfrak{U}_{E(\mathfrak{E}_{\Lambda^-})}.$$

On the other hand, since $T = \int \omega_\varphi(\mathfrak{z})\, d\mathfrak{z}$ by Theorem 29.3, we have

$$\mathfrak{U}_{E_T(\Lambda^-)} = \{\mathfrak{z} : \omega_\varphi(\mathfrak{z}) \in \Lambda^-\}^\circ$$

by the definition §23(1). Thus we obtain

$$E_T(\Lambda^-) \geqq E(\mathfrak{E}_{\Lambda^-})$$

for every bounded closed set Λ^- in G. For every bounded closed set Λ^-, we can find a sequence of bounded closed sets Λ_ν^- $(\nu = 1, 2, \ldots)$ in G such that

$$\Lambda^{-\prime} = \sum_{\nu=1}^{\infty} \Lambda_\nu^-,$$

and then we have hence by the postulate 3) in §28

$$E_T(\Lambda^{-\prime}) = \bigcup_{\nu=1}^{\infty} E_T(\Lambda_\nu^-) \geqq \bigcup_{\nu=1}^{\infty} E(\mathfrak{E}_{\Lambda_\nu^-}) = E(\mathfrak{E}_{\Lambda^{-\prime}}),$$

because $\mathfrak{E}_{\Lambda^{-\prime}} = \mathfrak{E}\sum_{\nu=1}^{\infty} \Lambda_\nu^- = \sum_{\nu=1}^{\infty} \mathfrak{E}_{\Lambda_\nu^-}$. Consequently we obtain

$$E_T(\Lambda^-) = 1 - E_T(\Lambda^{-\prime}) \leqq 1 - E(\mathfrak{E}_{\Lambda^{-\prime}}) = E(\mathfrak{E}_{\Lambda^-}),$$

because $\mathfrak{E}_{\Lambda^-}{}' = \mathfrak{E}_{\Lambda^{-\prime}}$. Therefore we have

$$E_T(\Lambda^-) = E(\mathfrak{E}_{\Lambda^-}) \qquad \text{for every bounded closed set } \Lambda.$$

For each open set Λ° in G, we can find a sequence of bounded closed sets Λ_ν^- $(\nu = 1, 2, \ldots)$ in G for which

$$\Lambda^\circ = \sum_{\nu=1}^{\infty} \Lambda_\nu^- .$$

Thus we obtain likewise

$$E_T(\Lambda^\circ) = E(\mathfrak{E}_{\Lambda^\circ}) \quad \text{for every open set } \Lambda^\circ .$$

From this relation we conclude easily that $E_T(\Lambda) = E(\mathfrak{E}_\Lambda)$ for every Borel set Λ in $\mathfrak{G}$.

Theorem 29.6. For two $(\mathfrak{m})$-measurable functions φ and ψ on Ω and for $\varepsilon > 0$, putting $\mathfrak{E}_\varepsilon = \{\zeta : |\varphi(\zeta) - \psi(\zeta)| < \varepsilon\}$,

$$\omega_\varphi(\mathfrak{x}) = \lim_{E\mathfrak{M}\to\mathfrak{x}} \varphi, \qquad \omega_\psi(\mathfrak{x}) = \lim_{E\mathfrak{M}\to\mathfrak{x}} \psi \qquad (\mathfrak{x} \in \mathfrak{X}_\mathfrak{R}),$$

we have that $|\omega_\varphi(\mathfrak{x}) - \omega_\psi(\mathfrak{x})| < \varepsilon$ implies $\mathfrak{x} \in \mathfrak{U}_{E(\mathfrak{E}_\varepsilon)}$.

Proof. If $\mathfrak{x} \bar{\in} \mathfrak{U}_{E(\mathfrak{E}_\varepsilon)}$, then $\mathfrak{x} \in \mathfrak{U}_{E(\mathfrak{E}_\varepsilon')}$, because we have by the postulate 2) in §28

$$\mathfrak{U}_{E(\mathfrak{E}_\varepsilon)'} = \mathfrak{U}_{1-E(\mathfrak{E}_\varepsilon)} = \mathfrak{U}_{E(\mathfrak{E}_\varepsilon')}.$$

Thus for any $\delta > 0$ we can find by definition $\mathfrak{E} \in \mathfrak{M}$ such that $\mathfrak{x} \in \mathfrak{U}_{E(\mathfrak{E})}$ and

$$|\varphi(\zeta) - \omega_\varphi(\mathfrak{x})| \leqq \delta, \quad |\psi(\zeta) - \omega_\psi(\mathfrak{x})| \leqq \delta \quad \text{for } \zeta \in \mathfrak{E}.$$

Since $E(\mathfrak{E})E(\mathfrak{E}_\delta') \neq 0$, $\mathfrak{E}_\varepsilon' = \{\zeta : |\varphi(\zeta) - \psi(\zeta)| \geqq \varepsilon\}$, we obtain hence

$$|\omega_\varphi(\mathfrak{x}) - \omega_\psi(\mathfrak{x})| \geqq \varepsilon - 2\delta.$$

Since $\delta > 0$ may be arbitrary, we have therefore

$$|\omega_\varphi(\mathfrak{x}) - \omega_\psi(\mathfrak{x})| \geqq \varepsilon,$$

contradicting the assumption: $|\omega_\varphi(\mathfrak{x}) - \omega_\psi(\mathfrak{x})| < \varepsilon$.

Theorem 29.7. For two $(\mathfrak{m})$-measurable functions φ and ψ on Ω and for $P \in \mathfrak{R}$, we have

$$\int \varphi(\zeta) E(d\zeta) P = \int \psi(\zeta) E(d\zeta) P$$

if and only if $E(\{\zeta : \varphi(\zeta) \neq \psi(\zeta)\})P = 0$.

Proof. Since $\varphi - \psi$ also is $(\mathfrak{m})$-measurable by Theorem 27.3. putting

$$\mathfrak{E}_0 = \{\zeta : \varphi(\zeta) = \psi(\zeta)\},$$

we see easily by definition that $\mathfrak{E}_0 \in \mathfrak{M}$ and

$$\omega_\varphi(\mathfrak{x}) = \omega_\psi(\mathfrak{x}) \quad \text{for every } \mathfrak{x} \in \mathfrak{U}_{E(\mathfrak{E}_0)}.$$

Thus we have by Theorem 17.4

$$\int \omega_\varphi(\mathfrak{x})d\mathfrak{x}\, E(\mathfrak{E}_0)x = \int_{E(\mathfrak{E}_0)} \omega_\varphi(\mathfrak{x})d\mathfrak{x}\, E(\mathfrak{E}_0)x$$
$$= \int_{E(\mathfrak{E}_0)} \omega_\psi(\mathfrak{x})d\mathfrak{x}\, E(\mathfrak{E}_0)x = \int \omega_\psi(\mathfrak{x})d\mathfrak{x}\, E(\mathfrak{E}_0)x,$$

and hence

$$\int \omega_\varphi(\mathfrak{x})d\mathfrak{x}\, E(\mathfrak{E}_0) = \int \omega_\psi(\mathfrak{x})d\mathfrak{x}\, E(\mathfrak{E}_0).$$

Therefore we obtain by Theorem 29.3

$$\int \varphi(\zeta)E(d\zeta)E(\mathfrak{E}_0) = \int \psi(\zeta)E(d\zeta)E(\mathfrak{E}_0).$$

If $E(\{\zeta : \varphi(\zeta) \neq \psi(\zeta)\})P = 0$, then we have by 2) in §28

$$E(\mathfrak{E}_0)P = P,$$

because $\mathfrak{E}_0 = \{\zeta : \varphi(\zeta) \neq \psi(\zeta)\}'$. Thus we have

$$\int \varphi(\zeta)E(d\zeta)P = \int \varphi(\zeta)E(d\zeta)E(\mathfrak{E}_0)P$$
$$= \int \psi(\zeta)E(d\zeta)E(\mathfrak{E}_0)P = \int \psi(\zeta)E(d\zeta)P.$$

Conversely, putting

$$T_1 = \int \varphi(\zeta)E(d\zeta), \quad T_2 = \int \psi(\zeta)E(d\zeta),$$

if $T_1 P = T_2 P$, then we have by §25(9)

$$(T_1, \mathfrak{x}) = (T_2, \mathfrak{x}) \qquad \text{for every } \mathfrak{x} \in \mathfrak{U}_P,$$

because $(P, \mathfrak{x}) = 1$ for $\mathfrak{x} \in \mathfrak{U}_P$ by §25(49). Thus we obtain by Theorem 29.3

$$\omega_\varphi(\mathfrak{x}) = \omega_\psi(\mathfrak{x}) \qquad \text{for every } \mathfrak{x} \in \mathfrak{U}_P.$$

Since $\mathfrak{E}_0 = \prod_{\nu=1}^{\infty} \{\zeta : |\varphi(\zeta) - \psi(\zeta)| < \frac{1}{\nu}\}$, we conclude thus

$$\mathfrak{U}_P \subset \mathfrak{U}_{E(\mathfrak{E}_0)}$$

by Theorems 29.6 and 15.8. Therefore we have $P = PE(\mathfrak{E}_0)$, that is, $E(\mathfrak{E}_0')P = 0$.

Theorem 29.8. A sequence of integral operators

$$T_\nu = \int \varphi_\nu(\zeta)E(d\zeta) \qquad (\nu = 1, 2, \ldots)$$

is D-convergent if and only if we can find $\mathfrak{E}_0 \in \mathfrak{M}$ such that $E(\mathfrak{E}_0) = 1$ and $\varphi_\nu(\zeta)$ $(\nu = 1, 2, \ldots)$ is convergent for every $\zeta \in \mathfrak{E}_0$, and then there exists a $(\mathfrak{M})$-measurable function φ on Ω for which

$$\lim_{\nu\to\infty} \varphi_\nu(\zeta) = \varphi(\zeta) \qquad \text{for every } \zeta \in \mathfrak{E}_0,$$
$$\text{D-}\lim_{\nu\to\infty} T_\nu = \int \varphi(\zeta)E(d\zeta).$$

Proof. If $\varphi_\nu(\zeta)$ $(\nu = 1, 2, \ldots)$ is convergent for every $\zeta \in \mathfrak{E}_0$ and $E(\mathfrak{E}_0) = 1$, then for the characteristic function $\chi_{\mathfrak{E}_0}$ of $\mathfrak{E}_0$, $\varphi_\nu(\zeta)\chi_{\mathfrak{E}_0}(\zeta)$ $(\nu = 1, 2, \ldots)$ is convergent for every $\zeta \in \Omega$, and hence, putting

$$\varphi(\zeta) = \lim_{\nu \to \infty} \varphi_\nu(\zeta)\chi_{\mathfrak{E}_0}(\zeta) \qquad (\zeta \in \Omega),$$

we obtain by Theorem 27.2 a $(\mathfrak{m})$-measurable function φ on Ω for which

$$\lim_{\nu \to \infty} \varphi_\nu(\zeta) = \varphi(\zeta) \qquad \text{for every } \zeta \in \mathfrak{E}_0.$$

Thus, putting

$$\mathfrak{E}_{\mu,\sigma} = \prod_{\nu \geq \mu} \{\zeta : |\varphi_\nu(\zeta) - \varphi(\zeta)| < \frac{1}{\sigma}\}$$

we have $\mathfrak{E}_{\mu,\sigma} \in \mathfrak{m}$ for $\mu, \sigma = 1, 2, \ldots$, $\mathfrak{E}_{1,\sigma} \subset \mathfrak{E}_{2,\sigma} \subset \cdots$, and

$$\mathfrak{E}_0 \subset \sum_{\mu=1}^{\infty} \mathfrak{E}_{\mu,\sigma} \qquad \text{for every } \sigma = 1, 2, \ldots.$$

This relation yields

$$E(\mathfrak{E}_{\mu,\sigma}) \uparrow_{\mu=1}^{\infty} 1 \qquad \text{for every } \sigma = 1, 2, \ldots.$$

Furthermore, putting

$$\mathfrak{F}_\mu = \{\zeta : |\varphi(\zeta)| < \mu\} \qquad (\mu = 1, 2, \ldots),$$

we have obviously $\mathfrak{F}_\mu \in \mathfrak{m}$ $(\mu = 1, 2, \ldots)$, $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$, and further $\Omega = \sum_{\mu=1}^{\infty} \mathfrak{F}_\mu$. Thus we have

$$E(\mathfrak{F}_\mu) \uparrow_{\mu=1}^{\infty} 1.$$

Putting

$$T = \int \varphi(\zeta) E(d\zeta),$$

we have by Theorem 29.3

$$(T, \mathfrak{f}) = \lim_{E\mathfrak{m} \to \mathfrak{f}} \varphi \qquad \text{for every } \mathfrak{f} \in \mathfrak{L}_{\mathfrak{R}},$$

and further by assumption

$$(T_\nu, \mathfrak{f}) = \lim_{E\mathfrak{m} \to \mathfrak{f}} \varphi_\nu \qquad \text{for } \mathfrak{f} \in \mathfrak{L}_{\mathfrak{R}}, \ \nu = 1, 2, \ldots.$$

Thus, if $\mathfrak{f} \in \bigcup_{E(\mathfrak{F}_\mu)E(\mathfrak{E}_{\mu,\sigma})}$, then we have

$$|(T_\nu, \mathfrak{f}) - (T, \mathfrak{f})| \leqq \frac{1}{\sigma} \qquad \text{for } \nu \geq \mu,$$

and hence for $\mathfrak{f} \in \prod_{\sigma=1}^{\infty} \sum_{\mu=1}^{\infty} \bigcup_{E(\mathfrak{F}_\mu)E(\mathfrak{E}_{\mu,\sigma})}$ we have

$$\lim_{\nu \to \infty} (T_\nu, \mathfrak{f}) = (T, \mathfrak{f}).$$

On the other hand we have by Theorem 15.7

$$\Big(\sum_{\mu=1}^{\infty} \bigcup_{E(\mathfrak{F}_\mu)E(\mathfrak{E}_{\mu,\sigma})}\Big)^- = \bigcup_{\mu=1}^{\infty}{}_{E(\mathfrak{F}_\mu)E(\mathfrak{E}_{\mu,\sigma})} = \mathfrak{L}_{\mathfrak{R}},$$

and hence $\sum_{\mu=1}^{\infty} \bigcup_{E(\mathfrak{F}_\mu)E(\mathfrak{E}_{\mu,\sigma})}$ is an open dense set for every $\sigma = 1, 2, \ldots$. Consequently

$$\{\prod_{\sigma=1}^{\infty}\sum_{\mu=1}^{\infty} U_{E(\mathfrak{F}_\mu)E(\mathfrak{F}_{\mu,\sigma})}\}^\circ$$

also is an open dense set by Theorem 15.11. Therefore we have by definition

$$\underset{\nu\to\infty}{\text{D-lim}}\, T_\nu = T.$$

Conversely we suppose that T_ν $(\nu = 1, 2,\ldots)$ is D-convergent. Then we can find by definition an open dense set A in $\mathfrak{L}_{\mathfrak{R}}$ such that $(T_\nu, \mathfrak{f})$ $(\nu = 1, 2,\ldots)$ is convergent for every $\mathfrak{f} \in A$. On the other hand, putting

$$\mathfrak{F}_{\nu,\mu,\sigma} = \{\zeta : |\varphi_\nu(\zeta) - \varphi_\mu(\zeta)| < \frac{1}{\sigma}\} \qquad (\nu, \mu, \sigma = 1, 2,\ldots),$$

$$\mathfrak{F}_0 = \prod_{\sigma=1}^{\infty}\sum_{\rho=1}^{\infty}\prod_{\nu,\mu\geqq\rho} \mathfrak{F}_{\nu,\mu,\sigma},$$

we see easily that $\mathfrak{F}_0 \in \mathfrak{M}$ and $\varphi_\nu(\zeta)$ $(\nu = 1, 2,\ldots)$ is convergent for every $\zeta \in \mathfrak{F}_0$. For any $\mathfrak{f} \in A$ and $\sigma = 1, 2,\ldots$, we can find ρ such that

$$|(T_\nu, \mathfrak{f}) - (T_\mu, \mathfrak{f})| < \frac{1}{\sigma} \qquad \text{for } \nu, \mu \geqq \rho,$$

and hence we have by Theorem 29.6

$$\mathfrak{f} \in U_{E(\mathfrak{F}_{\nu,\mu,\sigma})} \qquad \text{for } \nu, \mu \geqq \rho.$$

Thus we have

$$A \subset \sum_{\rho=1}^{\infty}\prod_{\nu,\mu\geqq\rho} U_{E(\mathfrak{F}_{\nu,\mu,\sigma})} \qquad \text{for every } \sigma = 1, 2,\ldots.$$

Putting $B = A(\sum_{\rho=1}^{\infty}(\prod_{\nu,\mu\geqq\rho} U_{E(\mathfrak{F}_{\nu,\mu,\sigma})})^\circ)^{-\prime}$, we obtain an open set B and we have

$$B(\prod_{\nu,\mu\geqq\rho} U_{E(\mathfrak{F}_{\nu,\mu,\sigma})})^\circ = 0 \qquad \text{for every } \rho = 1, 2,\ldots.$$

This relation yields

$$(B^- \prod_{\nu,\mu\geqq\rho} U_{E(\mathfrak{F}_{\nu,\mu,\sigma})})^\circ = 0.$$

Consequently we obtain by Theorem 15.10

$$(\sum_{\rho=1}^{\infty} B^- \prod_{\nu,\mu\geqq\rho} U_{E(\mathfrak{F}_{\nu,\mu,\sigma})})^{-\circ} = 0.$$

On the other hand we have

$$B \subset B^- A \subset \sum_{\rho=1}^{\infty} B^- \prod_{\nu,\mu\geqq\rho} U_{E(\mathfrak{F}_{\nu,\mu,\sigma})},$$

hence $B^{-\circ} = 0$. This relation yields $B = 0$, because we have ...sly $B = B^\circ \subset B^{-\circ}$. Therefore we have

$$\subset (\sum_{\rho=1}^{\infty}(\prod_{\nu,\mu\geqq\rho} U_{E(\mathfrak{F}_{\nu,\mu,\sigma})})^\circ)^- \qquad \text{for every } \sigma = 1, 2,\ldots,$$

$$A \subset \Big(\prod_{\sigma=1}^{\infty}\Big(\sum_{\rho=1}^{\infty}\Big(\prod_{\nu,\mu\geq\rho} U_{E(\mathfrak{E}_{\nu,\mu,\sigma})}\Big)^{\circ}\Big)^{-}\Big)^{\circ}.$$

Consequently we obtain by Theorems 15.7 and 15.8

$$\mathfrak{F}_{\mathcal{R}} = A^{-} = U_{E(\mathfrak{E}_0)},$$

and hence $E(\mathfrak{E}_0) = 1$.

Theorem 29.9. Let T be a dilatator by $\mathcal{R}$ and E_T the spectrality of T. If for any open set Λ° in the complex plane $\mathfrak{G}$ we can find $\mathfrak{E} \in \mathfrak{M}$ such that

$$E_T(\Lambda^{\circ}) = E(\mathfrak{E})$$

for a spectrality E on $\mathfrak{M}$, then there exists a $(\mathfrak{M})$-measurable function φ for which

$$T = \int \varphi(\zeta) E(d\zeta).$$

Proof. Putting

$$\Lambda_\nu = \{\zeta : |\zeta| \leq \nu\} \qquad (\nu = 1, 2, \ldots),$$

we obtain a sequence of bounded closed sets Λ_ν $(\nu = 1, 2, \ldots)$ in $\mathfrak{G}$ such that

$$\Lambda_1 \subset \Lambda_2 \subset \cdots, \qquad \sum_{\nu=1}^{\infty} \Lambda_\nu = \mathfrak{G}.$$

Thus we have $E_T(\Lambda_\nu) \uparrow_{\nu=1}^{\infty} 1$, and hence obviously by definition

$$\mathfrak{D}\text{-}\lim_{\nu\to\infty} T E_T(\Lambda_\nu) = T.$$

Furthermore we have by the formulas (9), (49) in §25 and (16) in §23

$$(T E_T(\Lambda_\nu), \mathfrak{f}) = (T, \mathfrak{f}) \in \Lambda_\nu \quad \text{for } \mathfrak{f} \in U_{E_T(\Lambda_\nu)}.$$

Recalling Theorem 21.9, we obtain hence for every $x \in R$

$$(\mathcal{R})\text{-}\lim_{\varepsilon\to 0} \sum_{\mu=1}^{\varkappa} (T, \mathfrak{f}_\mu) E_T(\mathfrak{E}_\mu) x = T E_T(\Lambda_\nu) x$$

for a finite number of Borel sets $\mathfrak{E}_\mu$ $(\mu = 1, 2, \ldots, \varkappa)$ in $\mathfrak{G}$ such that

$$\Lambda_\nu = \sum_{\mu=1}^{\varkappa} \mathfrak{E}_\mu, \quad \mathfrak{E}_\mu \mathfrak{E}_\rho = 0 \text{ for } \mu \neq \rho, \quad \sup_{\xi,\eta\in\mathfrak{E}_\mu} |\xi - \eta| \leq \varepsilon,$$

and for $\mathfrak{f}_\mu \in U_{E_T(\mathfrak{E}_\mu)}$ $(\mu = 1, 2, \ldots, \varkappa)$. Therefore we have by Theorem 26.6

$$\mathfrak{D}\text{-}\lim_{\varepsilon\to 0} \sum_{\mu=1}^{\varkappa} (T, \mathfrak{f}_\mu) E_T(\mathfrak{E}_\mu) = T E_T(\Lambda_\nu).$$

On the other hand, denoting by $\mathfrak{K}$ the totality of those point se $\mathfrak{E}$ in $\mathfrak{G}$ for which we can find $\mathfrak{F} \in \mathfrak{M}$ such that $E_T(\mathfrak{E}) = E(\mathfrak{F})$ see easily that $\mathfrak{K}$ is a totally additive set class. Furthermor

contains all open sets by assumption. Therefore for any Borel set $\mathfrak{E}$ in G we can find $\mathfrak{F} \in \mathfrak{M}$ for which $E_T(\mathfrak{E}) = E(\mathfrak{F})$. Thus we can find $\mathfrak{F}_\mu \in \mathfrak{M}$ $(\mu = 1, 2, \ldots, \varkappa)$ such that

$$E_T(\mathfrak{E}_\mu) = E(\mathfrak{F}_\mu) \qquad (\mu = 1, 2, \ldots, \varkappa).$$

Furthermore we can assume that $\mathfrak{F}_\mu \mathfrak{F}_\rho = 0$ for $\mu \neq \rho$, because we have $E_T(\mathfrak{E}_\mu) E_T(\mathfrak{E}_\rho) = 0$ for $\mu \neq \rho$. For such $\mathfrak{F}_\mu \in \mathfrak{M}$ $(\mu = 1, 2, \ldots, \varkappa)$, putting

$$\varphi(\zeta) = \begin{cases} (T, \mathfrak{f}_\mu) & \text{for } \zeta \in \mathfrak{F}_\mu, \\ 0 & \text{for } \zeta \in \overline{\sum_{\mu=1}^{\varkappa} \mathfrak{F}_\mu}, \end{cases}$$

we see easily by Theorem 27.1 that φ is a $(\mathfrak{M})$-measurable function on Ω and we have obviously by definition

$$\sum_{\mu=1}^{\varkappa} (T, \mathfrak{f}_\mu) E_T(\mathfrak{E}_\mu) = \int \varphi(\zeta) E(d\zeta).$$

Therefore we conclude by Theorem 29.8 that T is an integral operator by the spectrality E on $\mathfrak{M}$.

For the Borel class $\mathfrak{L}$ in the complex plane G, putting

$$\varphi(\zeta) = \zeta \qquad (\zeta \in G),$$

we obtain obviously a $(\mathfrak{L})$-measurable function φ on G. And we have

Theorem 29.10. *For the spectrality E_T of a dilatator T by $\mathfrak{R}$ we have*

$$T = \int \zeta \, E_T(d\zeta).$$

Proof. Putting

$$T_1 = \int \zeta \, E_T(d\zeta),$$

we have by Theorem 29.3

$$(T_1, \mathfrak{f}) = \lim_{E_T \mathfrak{L} \to \mathfrak{f}} \zeta \qquad \text{for every } \mathfrak{f} \in \mathfrak{L}_{\mathfrak{R}}.$$

Thus, if $(T_1, \mathfrak{f})$ is finite for a point $\mathfrak{f}$, then for any $\varepsilon > 0$ we can find a Borel set $\mathfrak{E} \in \mathfrak{L}$ for which $\mathfrak{f} \in \mathfrak{U}_{E_T(\mathfrak{E})}$ and

$$|\zeta - (T_1, \mathfrak{f})| \leqq \varepsilon \qquad \text{for every } \zeta \in \mathfrak{E}.$$

On the other hand, for such $\mathfrak{f}$ and $\mathfrak{E}$, we have $(T, \mathfrak{f}) \in \overline{\mathfrak{E}}$ by the formula §23(16), and hence we obtain

$$|(T, \mathfrak{f}) - (T_1, \mathfrak{f})| \leqq \varepsilon.$$

Since $\varepsilon > 0$ may be arbitrary, we conclude hence that

$$(T, \mathfrak{f}) = (T_1, \mathfrak{f}) \qquad \text{if } (T_1, \mathfrak{f}) \text{ is finite.}$$

Therefore we obtain $(T, \mathfrak{x}) = (T_1, \mathfrak{x})$ for every $\mathfrak{x} \in \mathcal{E}_{\mathcal{R}}$, because both sides are continuous over the whole space $\mathcal{E}_{\mathcal{R}}$. Thus we conclude $T = T_1$ by Theorem 24.1.

§30 Functions of Dilatators

Let Γ be a space. Considering every system of complex numbers $(\zeta_\gamma)_{\gamma\in\Gamma}$ as a point, we obtain a space, which will be called the power of the complex plane G and denoted by G^Γ. Every point set

$$\{(\zeta_\gamma)_{\gamma\in\Gamma} : \zeta_{\gamma_\nu} \in \Lambda_\nu \quad (\nu = 1, 2, \ldots, \varkappa)\}$$

for a finite number of elements $\gamma_\nu \in \Gamma$ and Borel sets Λ_ν in G ($\nu = 1, 2, \ldots, \varkappa$) is called an elementary set in the power G^Γ. The least totally additive set class containing all elementary sets in G^Γ is called the Borel class of G^Γ and denoted by $\mathcal{L}^\Gamma$. Every point set in the Borel class $\mathcal{L}^\Gamma$ is called a Borel set in G^Γ.

Let T_γ ($\gamma \in \Gamma$) be a system of dilatators by a closed ring of projection operators $\mathcal{R}$ containing 1, and E_γ the spectrality of T_γ for every $\gamma \in \Gamma$. Then there exists uniquelv a spectrality $E(\mathfrak{X})$ ($\mathfrak{X} \in \mathcal{L}^\Gamma$) in $\mathcal{R}$ such that

$$E(\{(\zeta_\gamma)_{\gamma\in\Gamma} : \zeta_{\gamma_0} \in \Lambda\}) = E_{\gamma_0}(\Lambda)$$

for every $\gamma_0 \in \Gamma$ and Borel set Λ in G.

In fact, denoting by $\mathcal{K}$ the totality of those point sets $\mathfrak{X}$ in G^Γ for which

$$\{\mathfrak{x} : (T_\gamma, \mathfrak{x})_{\gamma\in\Gamma} \in \mathfrak{X}\}^{-\circ} = \{\mathfrak{x} : (T_\gamma, \mathfrak{x})_{\gamma\in\Gamma} \in \mathfrak{X}\}^{\circ-},$$

we can prove likewise as in §23 that $\mathcal{K}$ is a totally additive set class and, putting

$$U_{E(\mathfrak{X})} = \{\mathfrak{x} : (T_\gamma, \mathfrak{x})_{\gamma\in\Gamma} \in \mathfrak{X}\}^{\circ-} \quad \text{for } \mathfrak{X} \in \mathcal{K},$$

we obtain a spectrality $E(\mathfrak{X})$ ($\mathfrak{X} \in \mathcal{K}$) in G^Γ. Furthermore for any Borel set Λ in G and $\gamma_0 \in \Gamma$ we have obviously by Theorem 23.1

$$\{\mathfrak{x} : (T_{\gamma_0}, \mathfrak{x}) \in \Lambda\}^{-\circ} = \{\mathfrak{x} : (T_{\gamma_0}, \mathfrak{x}) \in \Lambda\}^{\circ-} = U_{E_{\gamma_0}(\Lambda)},$$

and hence $E(\{(\zeta_\gamma)_{\gamma\in\Gamma} : \zeta_{\gamma_0} \in \Lambda\}) = E_{\gamma_0}(\Lambda)$. Thus we have $\mathcal{K} \supset \mathcal{L}^\Gamma$ and $E(\mathfrak{X})$ ($\mathfrak{X} \in \mathcal{L}^\Gamma$) satisfies our requirement.

Such a spectrality $E(\mathfrak{X})$ ($\mathfrak{X} \in \mathcal{L}^\Gamma$) is determined uniquely. Because for two spectralities $E_1(\mathfrak{X})$ and $E_2(\mathfrak{X})$ ($\mathfrak{X} \in \mathcal{L}^\Gamma$), denoting by $\mathcal{K}_0$ the totality of those $\mathfrak{X} \in \mathcal{L}^\Gamma$ for which

$$E_1(\mathfrak{X}) = E_2(\mathfrak{X}),$$

we see easily that $\mathfrak{K}_0$ is a totally additive set class. If

$$E_1(\{(\zeta_\gamma)_{\gamma\in\Gamma}:\ \zeta_{\gamma_0}\in\Lambda\}) = E_2(\{(\zeta_\gamma)_{\gamma\in\Gamma}:\ \zeta_{\gamma_0}\in\Lambda\}),$$

that is,

$$\mathfrak{K}_0 \ni \{(\zeta_\gamma)_{\gamma\in\Gamma}:\ \zeta_{\gamma_0}\in\Lambda\}$$

for every $\gamma_0\in\Gamma$ and Borel set Λ in $\mathfrak{G}$, then we also have by the postulate 1) in §28

$$\mathfrak{K}_0 \ni \{(\zeta_\gamma)_{\gamma\in\Gamma}:\ \zeta_{\gamma_\nu}\in\Lambda_\nu\ (\nu=1,2,\dots,\varkappa)\}$$
$$= \prod_{\nu=1}^{\varkappa}\{(\zeta_\gamma)_{\gamma\in\Gamma}:\ \zeta_{\gamma_\nu}\in\Lambda_\nu\}$$

for every finite number of elements $\gamma_\nu\in\Gamma$ and Borel sets Λ_ν in $\mathfrak{G}$ ($\nu=1,2,\dots,\varkappa$), and hence $\mathfrak{K}_0$ contains all elementary sets in $\mathfrak{G}^\Gamma$. Therefore we have $\mathfrak{K}_0 = \mathfrak{L}^\Gamma$ by definition.

This uniquely determined spectrality $E(\mathfrak{E})$ ($\mathfrak{E}\in\mathfrak{L}^\Gamma$) in the power $\mathfrak{G}^\Gamma$ is called the <u>spectrality</u> of a system of dilatators T_γ ($\gamma\in\Gamma$).

Concerning the spectrality $E(\mathfrak{E})$ ($\mathfrak{E}\in\mathfrak{L}^\Gamma$) of a system of dilatators T_γ ($\gamma\in\Gamma$), we have hence for every $\mathfrak{E}\in\mathfrak{L}^\Gamma$

$$(1)\qquad U_{E(\mathfrak{E})} = \{\mathfrak{x}:\ (T_\gamma\mathfrak{x})_{\gamma\in\Gamma}\in\mathfrak{E}\}^{0-}$$
$$= \{\mathfrak{x}:\ (T_\gamma\mathfrak{x})_{\gamma\in\Gamma}\in\mathfrak{E}\}^{-0}$$

For a subspace Δ of Γ, we obtain likewise a power $\mathfrak{G}^\Delta$ of the complex plane $\mathfrak{G}$ by Δ and the spectrality $E^\Delta(\mathfrak{E})$ ($\mathfrak{E}\in\mathfrak{L}^\Delta$) of the system of dilatators T_γ ($\gamma\in\Delta$) for the Borel class $\mathfrak{L}^\Delta$ in $\mathfrak{G}^\Delta$.

For a point set $\mathfrak{E}$ in $\mathfrak{G}^\Delta$, putting

$$\mathfrak{F} = \{(\zeta_\gamma)_{\gamma\in\Gamma}:\ (\zeta_\gamma)_{\gamma\in\Delta}\in\mathfrak{E}\},$$

we obtain a point set $\mathfrak{F}$ in $\mathfrak{G}^\Gamma$. This point set $\mathfrak{F}$ is called the <u>cylinder set</u> of $\mathfrak{E}$ in $\mathfrak{G}^\Gamma$ and denoted by $\mathfrak{E}^\Gamma$.

Concerning cylinder sets, we have obviously by definition

$$(2)\qquad \Big(\sum_{\lambda\in\Lambda}\mathfrak{E}_\lambda\Big)^\Gamma = \sum_{\lambda\in\Lambda}\mathfrak{E}_\lambda^\Gamma,$$

$$(3)\qquad \Big(\prod_{\lambda\in\Lambda}\mathfrak{E}_\lambda\Big)^\Gamma = \prod_{\lambda\in\Lambda}\mathfrak{E}_\lambda^\Gamma,$$

$$(4)\qquad \mathfrak{E}'^\Gamma = \mathfrak{E}^{\Gamma\prime}.$$

Furthermore we have

<u>Theorem</u> 30.1. <u>For a point set $\mathfrak{E}$ in $\mathfrak{G}^\Delta$, the cylinder set $\mathfrak{E}^\Gamma$ of $\mathfrak{E}$ in $\mathfrak{G}^\Gamma$ is a Borel set, if and only if $\mathfrak{E}$ is a Borel set in</u> $\mathfrak{G}^\Delta$,

and then we have

$$E^{\Delta}(\mathfrak{E}) = E(\mathfrak{E}^{\Gamma}) \qquad (\mathfrak{E} \in \mathcal{G}^{\Delta})$$

for the spectrality E of T_γ $(\gamma \in \Gamma)$ and the spectrality E^{Δ} of the subsystem T_γ $(\gamma \in \Delta)$.

Proof. Denoting by $\mathcal{K}$ the totality of those point sets $\mathfrak{X}$ in $\mathcal{G}^{\Delta}$ for which $\mathfrak{X}^{\Gamma} \in \mathcal{G}^{\Gamma}$, we see easily by (2) and (4) that $\mathcal{K}$ is a totally additive set class. Since we have obviously by definition

$$\{(\zeta_\gamma)_{\gamma\in\Delta}: \ \zeta_{\gamma_\nu} \in \Lambda_\nu \ \ (\nu = 1, 2, \ldots, \varkappa)\}^{\Gamma}$$
$$= \{(\zeta_\gamma)_{\gamma\in\Gamma}: \ \zeta_{\gamma_\nu} \in \Lambda_\nu \ \ (\nu = 1, 2, \ldots, \varkappa)\}$$

for every finite number of elements $\gamma_\nu \in \Delta$ and Borel sets Λ_ν in $\mathcal{G}$ $(\nu = 1, 2, \ldots, \varkappa)$, we conclude by definition that the cylinder set $\mathfrak{E}^{\Gamma}$ is an elementary set in $\mathcal{G}^{\Gamma}$ for every elementary set $\mathfrak{E}$ in $\mathcal{G}^{\Delta}$. Thus $\mathcal{K}$ contains all elementary sets, and hence $\mathcal{K} > \mathcal{G}^{\Delta}$ by definition.

Furthermore, putting

$$E_1(\mathfrak{E}) = E(\mathfrak{E}^{\Gamma}) \qquad \text{for every } \mathfrak{E} \in \mathcal{G}^{\Delta},$$

we see easily that $E_1(\mathfrak{E})$ $(\mathfrak{E} \in \mathcal{G}^{\Delta})$ is a spectrality in $\mathcal{G}^{\Delta}$ and for every $\gamma_0 \in \Delta$ and Borel set Λ in $\mathcal{G}$ we have

$$E_1(\{(\zeta_\gamma)_{\gamma\in\Delta}: \ \zeta_{\gamma_0} \in \Lambda\})$$
$$= E(\{(\zeta_\gamma)_{\gamma\in\Gamma}: \ \zeta_{\gamma_0} \in \Lambda\}) = E_{\gamma_0}(\Lambda).$$

Thus $E_1(\mathfrak{E})$ $(\mathfrak{E} \in \mathcal{G}^{\Delta})$ is the spectrality of the subsystem T_γ $(\gamma \in \Delta)$ by definition, and hence

$$E^{\Delta}(\mathfrak{E}) = E(\mathfrak{E}^{\Gamma}) \qquad \text{for every } \mathfrak{E} \in \mathcal{G}^{\Delta}.$$

For every point set $\mathfrak{F}$ in $\mathcal{G}^{\Gamma}$, putting

$$\mathfrak{F}^{\Delta} = \{(\zeta_\gamma)_{\gamma\in\Delta}: \ (\zeta_\gamma)_{\gamma\in\Gamma} \in \mathfrak{F}\},$$

we obtain a point set $\mathfrak{F}^{\Delta}$ in $\mathcal{G}^{\Delta}$. Then we have obviously

$$\Big(\sum_{\lambda\in\Lambda} \mathfrak{F}_\lambda\Big)^{\Delta} = \sum_{\lambda\in\Lambda} \mathfrak{F}_\lambda^{\Delta},$$

and

$$\{(\zeta_\gamma)_{\gamma\in\Gamma}: \ \zeta_{\gamma_\nu} \in \Lambda_\nu \ \ (\nu = 1, 2, \ldots, \varkappa)\}$$
$$= \{(\zeta_\gamma)_{\gamma\in\Delta}: \ \zeta_{\gamma_{\nu_\mu}} \in \Lambda_{\nu_\mu} \ \ (\mu = 1, 2, \ldots, \rho)\},$$

if $\gamma_{\nu_\mu} \in \Delta$ for $\mu = 1, 2, \ldots, \rho$ and $\gamma_\nu \bar{\in} \Delta$ for every $\nu \neq \nu_\mu$ $(\mu = 1, 2, \ldots, \rho)$. Thus $\mathfrak{F}^{\Delta}$ is an elementary set in $\mathcal{G}^{\Delta}$ for every elementary set $\mathfrak{F}$ in $\mathcal{G}^{\Gamma}$. Denoting by $\mathcal{K}_0$ the totality of those point sets $\mathfrak{F}$ in $\mathcal{G}^{\Gamma}$ for which $\mathfrak{F}^{\Delta}$ is a Borel set in $\mathcal{G}^{\Delta}$, we see easily that $\mathcal{K}_0$ is a totally additive set class in $\mathcal{G}^{\Gamma}$ and $\mathcal{K}_0$ contains all

elementary sets, as proved just above. Consequently $\mathcal{K}_0 > \mathcal{L}_T^\Gamma$ by definition. Thus for a point set $\mathfrak{E}$ in G^Δ, if $\mathfrak{E}^\Gamma \in \mathcal{L}_T^\Gamma$, then

$$\mathfrak{E} = (\mathfrak{E}^\Gamma)^\Delta \in \mathcal{L}_T^\Delta.$$

For a function φ on G^Δ, putting

$$\psi((\zeta_\gamma)_{\gamma\in\Gamma}) = \varphi((\zeta_\gamma)_{\gamma\in\Delta}) \quad \text{for every } (\zeta_\gamma)_{\gamma\in\Gamma},$$

we obtain a function ψ on G^Γ. This function ψ is called the cylinder function of φ on G^Γ and denoted by φ^Γ, that is,

$$(5) \qquad \varphi^\Gamma((\zeta_\gamma)_{\gamma\in\Gamma}) = \varphi((\zeta_\gamma)_{\gamma\in\Delta}), \quad \text{for every } (\zeta_\gamma)_{\gamma\in\Gamma}.$$

With this definition we have

Theorem 30.2. For a finite function φ on a power G^Δ of G, $\Delta \subset \Gamma$, the cylinder function φ^Γ on G^Γ is $(\mathcal{L}_T^\Gamma)$-measurable, if and only if φ is $(\mathcal{L}_T^\Delta)$-measurable, and then we have

$$\int \varphi(\zeta) E^\Delta(d\zeta) = \int \varphi^\Gamma(\zeta) E(d\zeta).$$

Proof. For every point set Λ in G, we have obviously by definition

$$\{(\zeta_\gamma)_{\gamma\in\Delta} : \varphi((\zeta_\gamma)_{\gamma\in\Delta}) \in \Lambda\}^\Gamma = \{(\zeta_\gamma)_{\gamma\in\Gamma} : \varphi^\Gamma((\zeta_\gamma)_{\gamma\in\Gamma}) \in \Lambda\}.$$

Thus we conclude by Theorem 30.1 that φ^Γ is $(\mathcal{L}_T^\Gamma)$-measurable if and only if φ is $(\mathcal{L}_T^\Delta)$-measurable.

If a $(\mathcal{L}_T^\Delta)$-measurable function φ on G_T^Δ is bounded in some Borel set $\mathfrak{E}_0 \in \mathcal{L}_T^\Delta$, then for any $\varepsilon > 0$ we can find a finite number of Borel sets $\mathfrak{E}_\nu \in \mathcal{L}_T^\Delta$ $(\nu = 1, 2, \ldots, \varkappa)$ such that

$$\mathfrak{E}_0 = \sum_{\nu=1}^{\varkappa} \mathfrak{E}_\nu, \quad \mathfrak{E}_\nu \mathfrak{E}_\mu = 0 \quad \text{for } \nu \neq \mu,$$

$$\sup_{\xi,\eta\in\mathfrak{E}_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \quad \text{for every } \nu = 1, 2, \ldots, \varkappa.$$

For such $\mathfrak{E}_\nu \in \mathcal{L}_T^\Delta$ $(\nu = 1, 2, \ldots, \varkappa)$, we have obviously by Theorem 30.1

$$\mathfrak{E}_0^\Gamma = \sum_{\nu=1}^{\varkappa} \mathfrak{E}_\nu^\Gamma, \quad \mathfrak{E}_\nu^\Gamma \mathfrak{E}_\mu^\Gamma = 0 \text{ for } \nu \neq \mu,$$

$$\mathfrak{E}_\nu^\Gamma \in \mathcal{L}_T^\Gamma, \quad \sup_{\xi,\eta\in\mathfrak{E}_\nu^\Gamma} |\varphi^\Gamma(\xi) - \varphi^\Gamma(\eta)| \leqq \varepsilon \quad (\nu = 1, 2, \ldots, \varkappa),$$

and for every $x \in R$

$$\sum_{\nu=1}^{\varkappa} \varphi((\zeta_{\nu,\gamma})_{\gamma\in\Delta}) E^\Delta(\mathfrak{E}_\nu)x = \sum_{\nu=1}^{\varkappa} \varphi^\Gamma((\zeta_{\nu,\gamma})_{\gamma\in\Gamma}) E(\mathfrak{E}_\nu^\Gamma)x$$

for $(\zeta_{\nu,\gamma})_{\gamma\in\Gamma} \in \mathfrak{E}_\nu^\Gamma$ $(\nu = 1, 2, \ldots, \varkappa)$. Thus we have by definition

$$\int_{\mathfrak{F}_0} \varphi(\zeta) E^{\Delta}(d\zeta)x = \int_{\mathfrak{F}_0^{\Gamma}} \varphi^{\Gamma}(\zeta) E(d\zeta)x$$

for every $x \in R$. Therefore we conclude easily by definition

$$\int \varphi(\zeta) E^{\Delta}(d\zeta) = \int \varphi^{\Gamma}(\zeta) E(d\zeta).$$

Especially, putting $\Delta = \{\gamma_0\}$ in Theorem 30.2, we see by definition that $E^{\{\gamma_0\}}$ is the spectrality E_{γ_0} of T_{γ_0}, and hence we obtain by Theorem 29.10

$$(6) \qquad T_{\gamma_0} = \int \zeta_{\gamma_0} E(d(\zeta_\gamma)_{\gamma\in\Gamma}) \quad \text{for every } \gamma_0 \in \Gamma.$$

Theorem 30.3. *For any Borel set $\mathfrak{Y}$ in the power G^{Γ} of G we can find a countable subset $\Delta \subset \Gamma$ such that $\mathfrak{Y} = \mathfrak{E}^{\Gamma}$ for some Borel set $\mathfrak{E}$ in the power G^{Δ} by Δ.*

Proof. Denoting by $\mathfrak{K}$ the totality of those point sets $\mathfrak{Y}$ in G^{Γ} for which we can find a countable subset $\Delta \subset \Gamma$ such that $\mathfrak{Y} = \mathfrak{E}^{\Gamma}$ for some Borel set $\mathfrak{E}$ in G^{Δ}. For a sequence $\mathfrak{Y}_\nu \in \mathfrak{K}$ ($\nu = 1, 2, \ldots$) we can find then a sequence of countable subsets $\Delta_\nu \subset \Gamma$ ($\nu = 1, 2, \ldots$) such that $\mathfrak{Y}_\nu = \mathfrak{E}_\nu^{\Gamma}$ for some Borel set $\mathfrak{E}_\nu$ in G^{Δ_ν} for every $\nu = 1, 2, \ldots$. Denoting by Δ the union $\sum_{\nu=1}^{\infty} \Delta_\nu$ and by $\mathfrak{E}_\nu^{\Delta}$ the cylinder set of $\mathfrak{E}_\nu$ in G^{Δ}, we have obviously

$$\mathfrak{Y}_\nu = (\mathfrak{E}_\nu^{\Delta})^{\Gamma} \qquad \text{for every } \nu = 1, 2, \ldots,$$

and hence by the formula (2)

$$\left(\sum_{\nu=1}^{\infty} \mathfrak{Y}_\nu\right) = \left(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu^{\Delta}\right)^{\Gamma}.$$

Furthermore Δ is obviously a countable set and $\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu^{\Delta}$ is by Theorem 20.1 a Borel set in G^{Δ}. Thus we see by the formula (4) that $\mathfrak{K}$ is a totally additive set class. Since $\mathfrak{K}$ contains obviously all elementary sets, we have $\mathfrak{K} \supset \mathfrak{L}^{\Gamma}$ by definition.

Theorem 30.4. *For any $(\mathfrak{L}^{\Gamma})$-measurable function φ on the power G^{Γ} of G we can find a countable subset $\Delta \subset \Gamma$ such that $\varphi = \psi^{\Gamma}$ for some $(\mathfrak{L}^{\Delta})$-measurable function ψ on the power G^{Δ} by Δ.*

Proof. Let α_ν ($\nu = 1, 2, \ldots$) be the totality of rational complex numbers. Putting

$$\Lambda_{\nu,\mu} = \left\{\zeta : |\zeta - \alpha_\nu| < \frac{1}{\mu}\right\} \qquad (\nu, \mu = 1, 2, \ldots).$$

we have obviously

$$\{(\zeta_\gamma)_{\gamma\in\Gamma}:\ \varphi(\zeta_\gamma)_{\gamma\in\Gamma}\in\Lambda_{\nu,\mu}\}\in\mathfrak{L}^\Gamma\quad(\nu,\mu=1,2,\dots).$$

By virtue of Theorem 30.3, for each $\Lambda_{\nu,\mu}$ $(\nu,\mu=1,2,\dots)$ we can find hence a countable subset $\Delta_{\nu,\mu}\subset\Gamma$ such that

$$\{(\zeta_\gamma)_{\gamma\in\Gamma}:\ \varphi(\zeta_\gamma)_{\gamma\in\Gamma}\in\Lambda_{\nu,\mu}\}=\mathfrak{E}_{\nu,\mu}^\Gamma\quad\text{for some }\mathfrak{E}_{\nu,\mu}\in\mathfrak{L}^{\Delta_{\nu,\mu}}.$$

For such $\Delta_{\nu,\mu}$ $(\nu,\mu=1,2,\dots)$, putting

$$\Delta=\sum_{\nu,\mu=1}^{\infty}\Delta_{\nu,\mu},$$

we obtain a countable subset $\Delta\subset\Gamma$ and

$$\{(\zeta_\gamma)_{\gamma\in\Gamma}:\ \varphi(\zeta_\gamma)_{\gamma\in\Gamma}\in\Lambda_{\nu,\mu}\}=(\mathfrak{E}_{\nu,\mu}^\Delta)^\Gamma,$$

for every $\nu,\ \mu=1,2,\dots$. If

$$\varphi(\xi_\gamma)_{\gamma\in\Gamma}\neq\varphi(\eta_\gamma)_{\gamma\in\Gamma}$$

for two points $(\xi_\gamma)_{\gamma\in\Gamma}$ and $(\eta_\gamma)_{\gamma\in\Gamma}$ of $\mathfrak{G}^\Gamma$, then we can find $\Lambda_{\nu,\mu}$ and $\Lambda_{\nu',\mu'}$ such that

$$\varphi(\xi_\gamma)_{\gamma\in\Gamma}\in\Lambda_{\nu,\mu},\quad\varphi(\eta_\gamma)_{\gamma\in\Gamma}\in\Lambda_{\nu',\mu'},\quad\Lambda_{\nu,\mu}\Lambda_{\nu',\mu'}=0,$$

and hence

$$(\xi_\gamma)_{\gamma\in\Gamma}\in(\mathfrak{E}_{\nu,\mu}^\Delta)^\Gamma,\ (\eta_\gamma)_{\gamma\in\Gamma}\in(\mathfrak{E}_{\nu',\mu'}^\Delta)^\Gamma,\ (\mathfrak{E}_{\nu,\mu}^\Delta)^\Gamma(\mathfrak{E}_{\nu',\mu'}^\Delta)^\Gamma=0.$$

Thus we have $\mathfrak{E}_{\nu,\mu}^\Delta\mathfrak{E}_{\nu',\mu'}^\Delta=0$, and consequently

$$(\xi_\gamma)_{\gamma\in\Delta}\neq(\eta_\gamma)_{\gamma\in\Delta}.$$

Therefore there exists a function ψ on $\mathfrak{G}^\Delta$ for which $\varphi=\psi^\Gamma$. Furthermore ψ is $(\mathfrak{L}^\Delta)$-measurable by Theorem 30.2.

For a $(\mathfrak{L}^\Gamma)$-measurable function φ on the power $\mathfrak{G}^\Gamma$ of $\mathfrak{G}$ and the spectrality $E(\mathfrak{E})$ $(\mathfrak{E}\in\mathfrak{L}^\Gamma)$ of a system of dilatators T_γ $(\gamma\in\Gamma)$ by $\mathfrak{R}$, the integral operator

$$\int\varphi(\zeta)E(d\zeta)$$

is called a <u>function</u> of T_γ $(\gamma\in\Gamma)$ and denoted by

$$\varphi(T_\gamma)_{\gamma\in\Gamma}.$$

With this definition we have obviously by Theorems 30.4 and 30.2

<u>Theorem</u> 30.5 <u>For any function of dilatators</u> $\varphi(T_\gamma)_{\gamma\in\Gamma}$ <u>we can find a sequence</u> $\gamma_\nu\in\Gamma$ $(\nu=1,2,\dots)$ <u>and a function</u>

$$\psi(T_{\gamma_\nu})_{\nu=1,2,\dots}$$

<u>such that</u> $\varphi(T_\gamma)_{\gamma\in\Gamma}=\psi(T_{\gamma_\nu})_{\nu=1,2,\dots}$.

By virtue of this Theorem 30.5, we need only consider functions of

countable dilatators. In the sequel we shall denote by $\mathfrak{G}^\omega$ the power of the complex plane $\mathfrak{G}$ by $\{1, 2, \dots\}$ and by $\mathfrak{L}^\omega$ the Borel class in $\mathfrak{G}^\omega$.

Theorem 30.6. *For a function $\varphi(T_\nu)_{\nu=1,2,\dots}$ of countable dilatators T_ν by $\mathfrak{R}$ $(\nu = 1, 2, \dots)$ we have*

$$\varphi(T_\nu)_{\nu=1,2,\dots} = \int \varphi(T_\nu, \mathfrak{x})_{\nu=1,2,\dots}\, d\mathfrak{x}.$$

Proof. Putting $T = \varphi(T_\nu)_{\nu=1,2,\dots}$, we have by Theorem 29.3

$$(T, \mathfrak{x}) = \lim_{E\mathfrak{L}^\omega \to \mathfrak{x}} \varphi$$

for the spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L}^\omega)$ of the system T_ν $(\nu = 1, 2, \dots)$. For any $\varepsilon > 0$ we can find by Theorem 27.1 a sequence $\mathfrak{E}_\mu \in \mathfrak{L}^\omega$ $(\mu = 1, 2, \dots)$ such that

$$\sum_{\mu=1}^{\infty} \mathfrak{E}_\mu = \mathfrak{G}^\omega, \quad \sup_{\xi, \eta \in \mathfrak{E}_\mu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \qquad (\mu = 1, 2, \dots).$$

Then we have by Theorem 29.2 for every $\mu = 1, 2, \dots$

$$\sup_{\zeta \in \mathfrak{E}_\mu, \mathfrak{x} \in \mathfrak{V}_{E(\mathfrak{E}_\mu)}} |\varphi(\zeta) - (T, \mathfrak{x})| \leqq \varepsilon.$$

Putting $A_\mu = \{\mathfrak{x} : (T_\nu, \mathfrak{x})_{\nu=1,2,\dots} \in \mathfrak{E}_\mu\}^\circ$ we have by (1)

$$A_\mu^- = \mathfrak{V}_{E(\mathfrak{E}_\mu)} \qquad (\mu = 1, 2, \dots),$$

and hence

$$|\varphi(T_\nu, \mathfrak{x})_{\nu=1,2,\dots} - (T, \mathfrak{x})| \leqq \varepsilon \quad \text{for every } \mathfrak{x} \in A_\mu.$$

Furthermore $\sum_{\mu=1}^{\infty} A_\mu$ is an open dense set in the proper space $\mathfrak{C}_\mathfrak{R}$, because

$$\Big(\sum_{\mu=1}^{\infty} A_\mu\Big)^- \supset \Big(\sum_{\mu=1}^{\infty} \mathfrak{V}_{E(\mathfrak{E}_\mu)}\Big)^- = \mathfrak{V}_{E(\sum_{\mu=1}^{\infty} \mathfrak{E}_\mu)} = \mathfrak{C}_\mathfrak{R}.$$

Therefore we can find a sequence of open dense sets B_μ $(\mu = 1, 2, \dots)$ in $\mathfrak{C}_\mathfrak{R}$ such that

$$|\varphi(T_\nu, \mathfrak{x})_{\nu=1,2,\dots} - (T, \mathfrak{x})| \leqq \frac{1}{\mu} \quad \text{for every } \mathfrak{x} \in B_\mu.$$

Then, putting $B = \big(\prod_{\mu=1}^{\infty} B_\mu\big)^\circ$, we obtain by Theorem 15.11 an open dense set B in $\mathfrak{C}_\mathfrak{R}$ and we have

$$\varphi(T_\nu, \mathfrak{x})_{\nu=1,2,\dots} = (T, \mathfrak{x}) \quad \text{for every } \mathfrak{x} \in B.$$

Therefore we obtain

$$\int \varphi(T_\nu, \mathfrak{x})_{\nu=1,2,\dots}\, d\mathfrak{x} = \int (T, \mathfrak{x})\, d\mathfrak{x} = T.$$

Theorem 30.7. *For a sequence of dilatators T_ν $(\nu = 1, 2, \dots)$ by $\mathfrak{R}$ there exists a bounded symmetric dilatator T by $\mathfrak{R}$ such that T*

is a function of the sequence T_ν $(\nu = 1, 2, \ldots)$ and every T_ν is a function of T.

Proof. For the spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L}^\omega)$ of the sequence T_ν $(\nu = 1, 2, \ldots)$, putting

$$\mathfrak{K} = \{ E(\mathfrak{E}) : \mathfrak{E} \in \mathfrak{L}^\omega \},$$

we see easily that $\mathfrak{K}$ is a ring of projection operators and that $P_\nu \in \mathfrak{K}$ $(\nu = 1, 2, \ldots)$ implies

$$\bigcup_{\nu=1}^{\infty} P_\nu, \quad \bigcap_{\nu=1}^{\infty} P_\nu \in \mathfrak{K}.$$

Furthermore for the spectrality E_ν of T_ν we have

$$\mathfrak{K} \ni E_\nu(\mathfrak{F}) \qquad \text{for every Borel set } \mathfrak{F} \text{ in } \mathfrak{G}$$

for every $\nu = 1, 2, \ldots$, because we have by definition

$$E_\nu(\mathfrak{F}) = E(\mathfrak{F}^\omega)$$

for the cylinder set $\mathfrak{F}^\omega$ of $\mathfrak{F}$ in $\mathfrak{G}^\omega$

We set

$$\Lambda_{\mu,\rho} = \{ \zeta : |\zeta - \alpha_\mu| < \tfrac{1}{\rho} \} \qquad (\mu, \rho = 1, 2, \ldots)$$

for the totality of rational complex numbers α_μ $(\mu = 1, 2, \ldots)$. Let

$$P_{\sigma,\lambda} \neq 0 \qquad (\lambda = 1, 2, \ldots, \varkappa_\sigma)$$

be the system of all minimal elements in the ring generated by 1 and $E_\nu(\Lambda_{\mu,\rho})$ for $\nu, \mu, \rho \leqq \sigma$. Then we have obviously

$$\sum_{\lambda=1}^{\varkappa_\sigma} P_{\sigma,\lambda} = 1, \quad P_{\sigma,\lambda} P_{\sigma,\lambda'} = 0 \qquad \text{for } \lambda \neq \lambda',$$

$$P_{\sigma,\lambda} E_\nu(\Lambda_{\mu,\rho}) = P_{\sigma,\lambda} \text{ or } 0 \qquad \text{for } \nu, \mu, \rho \leqq \sigma,$$

$$P_{\sigma,\lambda} P_{\sigma',\lambda'} = P_{\sigma',\lambda'} \text{ or } 0 \qquad \text{for } \sigma < \sigma',$$

and hence

$$E_\nu(\Lambda_{\mu,\rho}) = \sum_{E_\nu(\Lambda_{\mu,\rho}) \geqq P_{\sigma,\lambda}} P_{\sigma,\lambda} \qquad \text{for } \nu, \mu, \rho \leqq \sigma,$$

$$P_{\sigma,\lambda} = \sum_{P_{\sigma,\lambda} \geqq P_{\sigma',\lambda'}} P_{\sigma',\lambda'} \qquad \text{for } \sigma < \sigma'.$$

Furthermore we can find closed intervals

$$[\xi_{\sigma,\lambda}, \eta_{\sigma,\lambda}] \qquad (\sigma = 1, 2, \ldots; \ \lambda = 1, 2, \ldots, \varkappa_\sigma)$$

in the real axis of the complex plane $\mathfrak{G}$ such that

$$[\xi_{\sigma,\lambda}, \eta_{\sigma,\lambda}][\xi_{\sigma,\lambda'}, \eta_{\sigma,\lambda'}] = 0 \qquad \text{for } \lambda \neq \lambda',$$

$$\eta_{\sigma,\lambda} - \xi_{\sigma,\lambda} < \frac{1}{\sigma},$$

and that $P_{\sigma,\lambda} \geqq P_{\sigma',\lambda'}$ implies

$$[\xi_{\sigma,\lambda}, \eta_{\sigma,\lambda}] \supset [\xi_{\sigma',\lambda'}, \eta_{\sigma',\lambda'}]$$

For any point $\mathfrak{P} \in \mathfrak{C}_{\mathfrak{R}}$ we can find uniquely λ_σ $(\sigma = 1, 2, \ldots)$ such that

$$\mathfrak{P} \in U_{P_{\sigma,\lambda_\sigma}} \qquad \text{for every } \sigma = 1, 2, \ldots,$$

and hence $P_{\sigma,\lambda_\sigma} \downarrow_{\sigma=1}^{\infty}$. For such λ_σ $(\sigma = 1, 2, \ldots)$ we have obviously

$$\xi_{1,\lambda_1} \leqq \xi_{2,\lambda_2} \leqq \cdots, \qquad \eta_{1,\lambda_1} \geqq \eta_{2,\lambda_2} \geqq \cdots,$$

$$\lim_{\sigma\to\infty} \xi_{\sigma,\lambda_\sigma} = \lim_{\sigma\to\infty} \eta_{\sigma,\lambda_\sigma}.$$

Thus, putting

$$\varphi(\mathfrak{P}) = \lim_{\sigma\to\infty} \xi_{\sigma,\lambda_\sigma} = \lim_{\sigma\to\infty} \eta_{\sigma,\lambda_\sigma}$$

for $\mathfrak{P} \in U_{P_{\sigma,\lambda_\sigma}}$ $(\sigma = 1, 2, \ldots)$, we obtain a bounded real valued function φ on the proper space $\mathfrak{C}_{\mathfrak{R}}$. This function φ is continuous, because for any $\mathfrak{P}_0 \in \mathfrak{C}_{\mathfrak{R}}$ and $\sigma = 1, 2, \ldots$ we can find λ_σ such that

$$\mathfrak{P}_0 \in U_{P_{\sigma,\lambda_\sigma}},$$

$$\xi_{\sigma,\lambda_\sigma} \leqq \varphi(\mathfrak{P}) \leqq \eta_{\sigma,\lambda_\sigma} \qquad \text{for } \mathfrak{P} \in U_{P_{\sigma,\lambda_\sigma}},$$

$$\eta_{\sigma,\lambda_\sigma} - \xi_{\sigma,\lambda_\sigma} < \frac{1}{\sigma}.$$

For such φ, putting

$$T = \int \varphi(\mathfrak{P})\, d\mathfrak{P},$$

we obtain by Theorem 25.4 and the formula §25(12) a bounded symmetric dilatator T by $\mathfrak{R}$. For the spectrality E_T of T, we have by the definition §23(2) for every open set Λ° in G

$$U_{E_T(\Lambda^\circ)} = \Big(\sum_{\Lambda^\circ > [\xi_{\sigma,\lambda},\, \eta_{\sigma,\lambda}]} U_{P_{\sigma,\lambda}}\Big)^-,$$

and hence by Theorem 15.7

$$E_T(\Lambda^\circ) = \bigvee_{\Lambda^\circ > [\xi_{\sigma,\lambda},\, \eta_{\sigma,\lambda}]} P_{\sigma,\lambda}.$$

Thus we have $E_T(\Lambda^\circ) \in \mathfrak{K}$ for every open set Λ° in G. Therefore we conclude by Theorem 29.9 that T is a function of the sequence T_ν $(\nu = 1, 2, \ldots)$.

On the other hand, it is obvious that

$$E_T([\xi_{\sigma,\lambda}, \eta_{\sigma,\lambda}]) = P_{\sigma,\lambda} \qquad (\sigma = 1, 2, \ldots; \lambda = 1, 2, \ldots, \varkappa_\sigma)$$

and hence we have for every $\nu = 1, 2, \ldots$

$$E_\nu(\Lambda_{\mu,\rho}) = \sum_{E_\nu(\Lambda_{\mu,\rho}) \geqq P_{\sigma,\lambda}} P_{\sigma,\lambda}$$

$$= E_T\Big(\sum_{E_\nu(\Lambda_{\mu,\rho}) \geqq P_{\sigma,\lambda}} [\xi_{\sigma,\lambda}, \eta_{\sigma,\lambda}]\Big)$$

For any open set Λ° in G we can find μ_τ, ρ_τ $(\tau = 1, 2, \ldots)$ such

that $\Lambda^\circ = \sum_{\tau=1}^{\infty} \Lambda_{\mu_\tau, \rho_\tau}$, and then we have for each $\nu = 1, 2, \ldots$

$$E_\nu(\Lambda^\circ) = E_\tau\Big(\sum_{\tau=1}^{\infty} \sum_{E_\nu(\Lambda_{\mu_\tau,\rho_\tau}) \geq P_{\sigma,\lambda}} [\xi_{\sigma,\lambda}, \eta_{\sigma,\lambda}]\Big).$$

Therefore each T_ν is a function of T by Theorem 29.9.

For a system of dilatators T_γ $(\gamma \in \Gamma)$ by $\mathfrak{R}$, the closed ring generated by the spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L}^\Gamma)$ of T_γ $(\gamma \in \Gamma)$ is called the spectral ring of the system T_γ $(\gamma \in \Gamma)$. A system of dilatators T_γ $(\gamma \in \Gamma)$ by $\mathfrak{R}$ is said to be proper, if the spectral ring of T_γ $(\gamma \in \Gamma)$ coincides with $\mathfrak{R}$.

Theorem 30.6. If a system of dilatators T_γ $(\gamma \in \Gamma)$ by $\mathfrak{R}$ is proper and there exists $a \in R$ such that $Pa \neq 0$ for every $0 \neq P \in R$, then every dilatator by $\mathfrak{R}$ is a function of the system T_γ $(\gamma \in \Gamma)$.

Proof. We have obviously by assumption $C_{[a]} = 1$. Putting

$$\mathfrak{K} = \{ E(\mathfrak{E}) : \mathfrak{E} \in \mathfrak{L}^\Gamma \}$$

for the spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L}^\Gamma)$ of the system T_γ $(\gamma \in \Gamma)$, we see easily that $\mathfrak{K}$ is a ring containing 1 and that $P_\nu \in \mathfrak{K}$ $(\nu = 1, 2, \ldots)$ implies $\bigcup_{\nu=1}^{\infty} P_\nu \in \mathfrak{K}$. Furthermore $\mathfrak{K}$ is closed, because for any system $P_\lambda \in \mathfrak{K}$ $(\lambda \in \Lambda)$ we can find by Theorem 14.7 a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ such that

$$\bigcup_{\lambda \in \Lambda} P_\lambda C_{[a]} = \bigcup_{\nu=1}^{\infty} P_{\lambda_\nu} C_{[a]}.$$

Therefore we have $\mathfrak{K} = \mathfrak{R}$, because the system T_γ $(\gamma \in \Gamma)$ is proper by assumption.

For any dilatator T by $\mathfrak{R}$, denoting by E_T the spectrality of T, we conclude hence that for any Borel set Λ in G_T we can find $\mathfrak{E} \in \mathfrak{L}^\Gamma$ such that

$$E_T(\Lambda) = E(\mathfrak{E}).$$

Therefore T is a function of the system T_γ $(\gamma \in \Gamma)$ by Theorem 29.9.

CHAPTER IV

NORMAL OPERATORS

§31 Fundamental Definitions

Let R be a Hilbert space. A linear operator N with domain D in R is said to be normal, if D is dense in R and there exists an operator $\bar{N}$ with the same domain D such that

1) $(Nx, y) = (x, \bar{N}y)$ for every $x, y \in D$,

2) $\|Nx\| = \|\bar{N}x\|$ for every $x \in D$.

Such an operator $\bar{N}$ is called the conjugate operator of N. Since D is dense in R, we see immediately by the postulate 1) that the conjugate operator of N is determined uniquely. Furthermore we see easily by definition that the conjugate operator $\bar{N}$ of a normal operator N also is normal and the conjugate operator of $\bar{N}$ coincides with N.

Let N be a normal operator with domain D and $\bar{N}$ the conjugate operator of N in the sequel. For every $x, y \in D$ and complex number λ we have by the postulate 2)

$$(N(x + \lambda y), N(x + \lambda y)) = (\bar{N}(x + \lambda y), \bar{N}(x + \lambda y)),$$

and hence by 2)

$$\mathcal{R}\,\bar{\lambda}(Nx, Ny) = \mathcal{R}\,\bar{\lambda}(\bar{N}x, \bar{N}y).$$

Thus, putting $\lambda = 1$ and i, we conclude from this relation

(3) $(Nx, Ny) = (\bar{N}x, \bar{N}y)$ for every $x, y \in D$.

For a sequence $a_\nu \in D$ $(\nu = 1, 2, \ldots)$, if Na_ν $(\nu = 1, 2, \ldots)$ is convergent, then $\bar{N}a_\nu$ $(\nu = 1, 2, \ldots)$ also is convergent, because we have by 2)

$$\|Na_\nu - Na_\mu\| = \|\bar{N}a_\nu - \bar{N}a_\mu\| \qquad (\nu, \mu = 1, 2, \ldots).$$

Furthermore, if $\lim_{\nu \to \infty} a_\nu = 0$ and Na_ν $(\nu = 1, 2, \ldots)$ is convergent, then for every $x \in D$ we have by 1)

$$\lim_{\nu \to \infty} (Na_\nu, x) = \lim_{\nu \to \infty} (a_\nu, \bar{N}x) = 0,$$

$$\lim_{\nu \to \infty} (\bar{N}a_\nu, x) = \lim_{\nu \to \infty} (a_\nu, Nx) = 0,$$

and hence

$$\lim_{\nu \to \infty} Na_\nu = \lim_{\nu \to \infty} \bar{N}a_\nu = 0,$$

because D is dense in R by definition. Thus, if

$$\lim_{\nu\to\infty} a_\nu = \lim_{\nu\to\infty} b_\nu, \qquad a_\nu, b_\nu \in D \ (\nu = 1, 2, \ldots),$$

and two sequences Na_ν and Nb_ν $(\nu = 1, 2, \ldots)$ are convergent, then

$$\lim_{\nu\to\infty} Na_\nu = \lim_{\nu\to\infty} Nb_\nu, \quad \lim_{\nu\to\infty} \bar{N}a_\nu = \lim_{\nu\to\infty} \bar{N}b_\nu.$$

Therefore, denoting by D_0 the totality of those elements $a \in R$ for which we can find a sequence $a_\nu \in D$ $(\nu = 1, 2, \ldots)$ such that Na_ν $(\nu = 1, 2, \ldots)$ is convergent and $\lim_{\nu\to\infty} a_\nu = a$, and putting

$$N_0 a = \lim_{\nu\to\infty} Na_\nu, \qquad \bar{N}_0 a = \lim_{\nu\to\infty} \bar{N}a_\nu$$

for such $a_\nu \in D$ $(\nu = 1, 2, \ldots)$, we obtain uniquely two operators N_0 and $\bar{N}_0$ with domain D_0 such that

$$N_0 x = Nx, \qquad \bar{N}_0 x = \bar{N}x \quad \text{for every } x \in D,$$

and for any $a \in D_0$ we can find a sequence $a_\nu \in D$ $(\nu = 1, 2, \ldots)$ for which

$$\lim_{\nu\to\infty} a_\nu = a, \quad \lim_{\nu\to\infty} Na_\nu = N_0 a, \quad \lim_{\nu\to\infty} \bar{N}a_\nu = \bar{N}_0 a.$$

We see easily by definition that such N_0 is a normal operator and $\bar{N}_0$ is the conjugate operator of N_0. Furthermore both N_0 and $\bar{N}_0$ are closed. In fact, if

$$\lim_{\nu\to\infty} b_\nu = b, \qquad \lim_{\nu\to\infty} N_0 b_\nu = C$$

for a sequence $b_\nu \in D_0$ $(\nu = 1, 2, \ldots)$, then we can find a sequence of elements $a_\nu \in D$ $(\nu = 1, 2, \ldots)$ such that

$$\| a_\nu - b_\nu \| < \frac{1}{2^\nu}, \qquad \| Na_\nu - N_0 b_\nu \| < \frac{1}{2^\nu}.$$

This relation yields $\lim_{\nu\to\infty} a_\nu = b$, $\lim_{\nu\to\infty} Na_\nu = C$, and hence we obtain $b \in D_0$ and $N_0 b = C$. Thus N_0 is closed. We also can prove likewise that $\bar{N}_0$ is closed. This closed normal operator N_0 is called the <u>closure</u> of N, that is, a closed normal operator N_0 with domain D_0 is said to be a <u>closure</u> of a normal operator N with domain D, if $D_0 \supset D$, $N_0 x = Nx$ for every $x \in D$, and for any $a \in D_0$ we can find a sequence $a_\nu \in D$ $(\nu = 1, 2, \ldots)$ such that

$$\lim_{\nu\to\infty} a_\nu = a, \qquad \lim_{\nu\to\infty} Na_\nu = N_0 a.$$

With this definition it is evident that the closure of a normal operator is determined uniquely. Thus we have proved that for any normal operator N there exists uniquely the closure N_0 of N. Furhtermore we see easily that the closure of the conjugate operator $\bar{N}$ of N coincides with the conjugate operator $\bar{N}_0$ of the closure N_0 of N.

If a normal operator N is bounded, then we see easily that the

closure of N also is bounded and defined over the whole space R.

Conversely we have

Theorem 31.1. *If a normal operator N is defined over the whole space R, then both N and the conjugate operator $\bar{N}$ of N are bounded, and $\bar{N}$ coincides with the adjoint operator N^* of N.*

Proof. If N is not bounded, then we can find a sequence $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ such that $\|a_\nu\| \leqq 1$ but $\|Na_\nu\| \geqq \nu^2$ for every $\nu = 1, 2, \ldots$, and we have for every $x \in R$

$$\lim_{\nu\to\infty} \left(\frac{1}{\nu} Na_\nu, x\right) = \lim_{\nu\to\infty} \left(\frac{1}{\nu} a_\nu, \bar{N}x\right) = 0.$$

This relation yields by Theorem 9.6 $\sup_{\nu \geqq 1} \left\| \frac{1}{\nu} Na_\nu \right\| < +\infty$, contradicting $\frac{1}{\nu}\|Na_\nu\| \geqq \nu$ for every $\nu = 1, 2, \ldots$. Therefore N is bounded. It is evident by 1) that the conjugate operator $\bar{N}$ of N is the adjoint operator of N. Consequently $\bar{N}$ also is bounded by §10(2).

Theorem 31.2. *If a projection operator P is commutative to N, that is, if $PN \subset NP$, then NP also is a normal operator, $P\bar{N} \subset \bar{N}P$, and $\bar{N}P$ is the conjugate operator of NP.*

Proof. If $PN \subset NP$, then for every $x, y \in D$ we have by the postulate 1)

$$(P\bar{N}x, y) = (x, NPy) = (x, PNy) = (\bar{N}Px, y),$$

and hence $P\bar{N}x = \bar{N}Px$ for every $x \in D$, that is, $P\bar{N} \subset \bar{N}P$.

If NPx has any sense, that is, if $Px \in D$, then we have for every $y \in R$, subject to $Py \in D$,

$$(NPx, y) = (PNPx, y) = (x, P\bar{N}Py) = (x, \bar{N}Py),$$

and it is evident by 2) that

$$\|NPx\| = \|\bar{N}Px\|.$$

Therefore NP is a normal operator and $\bar{N}P$ is the conjugate operator of NP.

Theorem 31.3. *If a projection operator P is commutative to N then for the closure N_0 of N we have that $PN_0 \subset N_0P$ and N_0P is the closure of NP.*

Proof. If $N_0 a$ has any sense, then we can find by definition a sequence $a_\nu \in D$ $(\nu = 1, 2, \ldots)$ such that

$$\lim_{\nu\to\infty} a_\nu = a, \qquad \lim_{\nu\to\infty} N a_\nu = N_0 a,$$

and hence

$$\lim_{\nu\to\infty} P a_\nu = Pa, \quad \lim_{\nu\to\infty} N P a_\nu = \lim_{\nu\to\infty} P N a_\nu = P N_0 a,$$

because $PN \subset NP$ by assumption. Therefore we obtain by definition $N_0 P a = P N_0 a$. Consequently we have $P N_0 \subset N_0 P$.

If $\lim_{\nu\to\infty} a_\nu = a$, $\lim_{\nu\to\infty} NP a_\nu = \mathfrak{f}$, then $\lim_{\nu\to\infty} P a_\nu = Pa$, and hence $N_0 P a = \mathfrak{f}$ by definition. Furthermore, if $N_0 P a$ has any sense, then we can find by definition a sequence $a_\nu \in D$ $(\nu = 1, 2, \ldots)$ such that

$$\lim_{\nu\to\infty} a_\nu = Pa, \qquad \lim_{\nu\to\infty} N a_\nu = N_0 P a.$$

From this relation we conclude

$$\lim_{\nu\to\infty} \{ P a_\nu + (1 - P) a \} = a,$$

$$\lim_{\nu\to\infty} N P \{ P a_\nu + (1 - P) a \} = \lim_{\nu\to\infty} N P a_\nu$$

$$= \lim_{\nu\to\infty} P N a_\nu = P N_0 P a = N_0 P a,$$

because $P N_0 \subset N_0 P$ as proved just above. Therefore $N_0 P$ is the closure of NP by definition.

For a linear operator T with domain D the manifold

$$\{ x : \ Tx = 0, \ x \in D. \}$$

is called the zero manifold of T. For a normal operator N, we see immediatley by 2) that the zero manifold of N coincides with that of the conjugate operator $\bar{N}$.

Theorem 31.4. If N is closed, then the zero manifold $\mathfrak{Z}$ of N is a closed linear manifold; the projection operator $P_{\mathfrak{Z}}$ of $\mathfrak{Z}$ is commutative to N; and every projection operator, which is commutative to N, also is commutative to $P_{\mathfrak{Z}}$.

Proof. It is evident by definition that the zero manifold $\mathfrak{Z}$ of N is a closed linear manifold, and we have obviously

$$N P_{\mathfrak{Z}} x = 0 \qquad \text{for every } x \in R.$$

Thus for every $a \in D$ we have by 1) and 2)

$$(P_{\mathfrak{Z}} N a, x) = (a, \bar{N} P_{\mathfrak{Z}} x) = 0 \qquad \text{for every } x \in R,$$

and hence $P_{\mathfrak{Z}} N a = 0$ for every $a \in D$. Therefore $P_{\mathfrak{Z}} N \subset N P_{\mathfrak{Z}}$ by definition.

If a projection operator Q is commutative to N, then we have

$$N Q P_Z x = Q N P_Z x = 0 \qquad \text{for every } x \in R,$$

and hence $Q P_Z x \in Z$ for every $x \in R$. Therefore we obtain by Theorem 12.6 $Q P_Z = P_Z Q$.

Theorem 31.5. If N is closed, then, denoting by M the totality of those elements $a \in D$ for which

$$N^\nu \bar{N}^\mu a \in D \qquad \text{for } \nu, \mu = 0, 1, 2, \ldots,$$

$$\sup_{\nu \geq 1} \| N^\nu \bar{N}^\mu a \| < +\infty \qquad \text{for } \mu = 0, 1, 2, \ldots,$$

we obtain a closed linear manifold M, and we have that $a \in M$ is equivalent to

$$\| N^\nu \bar{N}^\mu a \| \leqq \| a \| \qquad \text{for } \nu, \mu = 0, 1, 2, \ldots .$$

The projection operator P_M of M is commutative to N, and every projection operator, which is commutative to N, also is commutative to P_M.

Proof. Let M_0 be the totality of those elements $a \in D$ for which $N^\nu a \in D$ for every $\nu = 1, 2, \ldots$ and

$$\sup_{\nu \geq 1} \| N^\nu a \| < +\infty .$$

For any x, $y \in M_0$, we have

$$\| N^\nu(\alpha x + \beta y) \| \leqq |\alpha| \| N^\nu x \| + |\beta| \| N^\nu y \|$$

for every $\nu = 1, 2, \ldots$ and complex numbers α, β, and hence we see that M_0 is a linear manifold. For each $x \in M_0$, we have by 1) and 2) for every $\nu = 1, 2, \ldots$

$$\| N^\nu x \|^2 = (N^\nu x, N^\nu x) = (\bar{N} N^\nu x, N^{\nu-1} x)$$
$$\leqq \| \bar{N} N^\nu x \| \, \| N^{\nu-1} x \| = \| N^{\nu+1} x \| \, \| N^{\nu-1} x \| .$$

Thus, if $N^\nu x = 0$ for some ν, then $N^\nu x = 0$ for every $\nu = 1, 2, \ldots$. If $N x \neq 0$, then we have hence $N^\nu x \neq 0$ for every $\nu = 1, 2, \ldots$ and

$$\frac{\| N^\nu x \|}{\| N^{\nu-1} x \|} \leqq \frac{\| N^{\nu+1} x \|}{\| N^\nu x \|} \qquad \text{for } \nu = 1, 2, \ldots .$$

Since $\sup_{\nu \geq 1} \| N^\nu x \| < +\infty$ by assumption, we obtain hence

$$\frac{\| N^\nu x \|}{\| N^{\nu-1} x \|} \leqq 1 \qquad \text{for } \nu = 1, 2, \ldots,$$

and consequently

$$\| N^\nu x \| \leqq \| x \| \qquad \text{for every } \nu = 1, 2, \ldots .$$

Conversely if $\|N^\nu x\| \leqq \|x\|$ for every $\nu = 1, 2, \ldots$, then we have obviously $x \in M_0$. Since M_0 is a linear manifold, if

$$\lim_{\mu \to \infty} x_\mu = x, \qquad x_\mu \in M_0 \qquad (\mu = 1, 2, \ldots),$$

then we have $x_\mu - x_\rho \in M_0$ $(\mu, \rho = 1, 2, \ldots)$ and hence

$$\|N^\nu x_\mu - N^\nu x_\rho\| \leqq \|x_\mu - x_\rho\|$$

for every $\nu, \mu, \rho = 1, 2, \ldots$. Thus we see that $N^\nu x_\mu$ $(\mu = 1, 2, \ldots)$ is convergent for every $\nu = 1, 2, \ldots$. Since N is closed by assumption, we conclude hence

$$N^\nu x \in D, \qquad \|N^\nu x\| \leqq \|x\| \quad \text{for every } \nu = 1, 2, \ldots.$$

Consequently we obtain $x \in M_0$. Therefore M_0 is closed.

Putting

$$M = \{ x : \bar{N}^\mu x \in M_0 \text{ for every } \mu = 0, 1, 2, \ldots \},$$

we see easily that M is a linear manifold. If

$$\lim_{\rho \to \infty} x_\rho = x_0, \qquad x_\rho \in M \qquad (\rho = 1, 2, \ldots),$$

then we have $x_0 \in M_0$, $x_\rho - x_0 \in M_0$ $(\rho = 1, 2, \ldots)$, and hence

$$\|\bar{N} x_\rho - \bar{N} x_0\| = \|N(x_\rho - x_0)\| \leqq \|x_\rho - x_0\|$$

for every $\rho = 1, 2, \ldots$. This relation yields

$$\lim_{\rho \to \infty} \bar{N} x_\rho = \bar{N} x_0.$$

Since $\bar{N} x_\rho \in M_0$ for every $\rho = 1, 2, \ldots$, we obtain hence $\bar{N} x_0 \in M_0$ and it is evident that $\bar{N} x_\rho \in M$ for every $\rho = 1, 2, \ldots$. Thus we conclude that $\bar{N}^\mu x_0 \in M_0$ for every $\mu = 1, 2, \ldots$, that is, we have $x_0 \in M$. Therefore M also is closed.

Since $x \in M$ implies obviously $\bar{N} x \in M$, for the projection operator P_M of M we have $P_M \bar{N} P_M = \bar{N} P_M$. Thus we have for every $x, y \in D$

$$(P_M N x, y) = (x, \bar{N} P_M y) = (x, P_M \bar{N} P_M y) = (P_M N P_M x, y),$$

and hence $P_M N x = P_M N P_M x$ for every $x \in D$. On the other hand, for every $x \in M$ we have obviously $N x \in M_0$, and hence by 1) and (3) for every $y \in D$

$$(\bar{N} N x, y) = (N x, N y) = (\bar{N} x, \bar{N} y) = (N \bar{N} x, y).$$

Thus we have $\bar{N} N x = N \bar{N} x \in M_0$, because $\bar{N} x \in M$. We can prove likewise further by induction

$$\bar{N}^\mu N x = N \bar{N}^\mu x \in M_0 \quad \text{for every } \mu = 1, 2, \ldots,$$

and consequently $N x \in M$. Therefore we have

$$P_M N P_M = N P_M.$$

Consequently $P_M N x = N P_M x$ for every $x \in D$, that is, we have

For a projection operator Q, if $QN \subset NQ$, then we also have $Q\bar{N} \subset \bar{N}Q$ by Theorem 31.2. Thus for any $x \in M$ we have

$$\| N^{\nu} \bar{N}^{\mu} Q x \| = \| Q N^{\nu} \bar{N}^{\mu} x \| \leqq \| N^{\nu} \bar{N}^{\mu} x \| \leqq \| \bar{N}^{\mu} x \|$$

for every ν, $\mu = 0, 1, 2, \ldots$, and consequently $Qx \in M$. Therefore we obtain $Q P_M = P_M Q$ by Theorem 12.6.

If $x \in M$, then we have $\bar{N}^{\mu} x \in M_0$ for every $\mu = 1, 2, \ldots$, and hence

$$\| N^{\nu} N^{\mu} x \| \leqq \| \bar{N}^{\mu} x \| \quad \text{for every } \nu = 1, 2, \ldots .$$

On the other hand we have by 2)

$$\| \bar{N}^{\mu} x \| = \| N \bar{N}^{\mu-1} x \| \leqq \| \bar{N}^{\mu-1} x \|$$

for every $\mu = 1, 2, \ldots$, and hence $\| \bar{N}^{\mu} x \| \leqq \| x \|$ for every $\mu = 1, 2, \ldots$. Thus we obtain

$$\| N^{\nu} \bar{N}^{\mu} x \| \leqq \| x \| \quad \text{for every } \nu, \mu = 0, 1, 2, \ldots .$$

Conversely if $\| N^{\nu} \bar{N}^{\mu} x \| \leqq \| x \|$ for every ν, $\mu = 0, 1, 2, \ldots$, then we have $\bar{N}^{\mu} x \in M_0$ for every $\mu = 0, 1, 2, \ldots$, and hence $x \in M$.

§32 Regular Part

A normal operator N with domain D in R is said to be regular, if we can find a sequence of projection operators $P_{\nu} \uparrow_{\nu=1}^{\infty} 1$ such that $P_{\nu} N \subset N P_{\nu}$ and $N P_{\nu}$ is bounded for every $\nu = 1, 2, \ldots$.

With this definition we have obviously by Theorem 31.3

Theorem 32.1. *If a normal operator N is regular, then the closure of N also is regular.*

Furthermore we obtain by Theorem 31.2 and 2) in §31

Theorem 32.2. *If a normal operator N is regular, then the conjugate operator $\bar{N}$ of N also is regular.*

We have obviously by definition

Theorem 32.3. *Every bounded normal operator is regular.*

Theorem 32.4. *If a normal operator N is closed and regular, then we can find a sequence of projection operators $P_{\nu} \uparrow_{\nu=1}^{\infty} 1$ such that $P_{\nu} N \subset N P_{\nu}$, $N P_{\nu}$ is bounded for every $\nu = 1, 2, \ldots$, and $P_1 R$ includes the zero manifold of N.*

Proof. If a normal operator N is regular, then we can find by definition a sequence of projection operators $P_\nu \uparrow_{\nu=1}^{\infty} 1$ such that $P_\nu N \subset N P_\nu$, and $N P_\nu$ is bounded for every $\nu = 1, 2, \ldots$. If N is furthermore closed, then we see by Theorem 31.4 that the zero manifold Z of N is a closed linear manifold, and for the projection operator P_Z of Z we have

$$P_Z N \subset N P_Z, \quad P_\nu P_Z = P_Z P_\nu \qquad (\nu = 1, 2, \ldots).$$

Thus we have by Theorem 12.6

$$P_\nu \vee P_Z = P_\nu + P_Z - P_\nu P_Z \qquad (\nu = 1, 2, \ldots),$$

and hence we see easily that

$$(P_\nu \vee P_Z) N \subset N(P_\nu \vee P_Z),$$

$$N(P_\nu \vee P_Z) x = N P_\nu x \quad \text{for every } x \in D.$$

Furthermore it is evident that $P_\nu \vee P_Z \uparrow_{\nu=1}^{\infty} 1$.

For a projection operator Q, a normal operator N is said to be singular in Q, if $QN \subset NQ$ and $QP = 0$ for every projection operator P subject to the condition: $PN \subset NP$ and NP is bounded. A normal operator N is said to be singular, if N is singular in 1.

With this definition we have obviously by Theorem 31.2 and 2) in §31

Theorem 32.5. If a normal operator N is singular in a projection operator Q, then the conjugate operator $\bar{N}$ of N also is so in Q.

Let N be a closed normal operator with domain D in the sequel. We see easily by definition that αN also is a closed normal operator for every complex number $\alpha \neq 0$, and $\bar{\alpha}\bar{N}$ is the conjugate operator of αN. Thus, denoting by M_ρ $(\rho = 1, 2, \ldots)$ the totality of those elements $\alpha \in D$ for which

$$\left(\frac{1}{\rho} N\right)^\nu \left(\frac{1}{\rho} \bar{N}\right)^\mu \alpha \in D \qquad \text{for every } \nu, \mu = 0, 1, 2, \ldots,$$

$$\sup_{\nu \geq 1} \left\| \left(\frac{1}{\rho} N\right)^\nu \left(\frac{1}{\rho} \bar{N}\right)^\mu \alpha \right\| < +\infty \quad \text{for every } \mu = 0, 1, 2, \ldots,$$

we obtain by Theorem 31.5 a closed linear manifold M_ρ, and denoting by P_ρ the projection operator of M_ρ $(\rho = 1, 2, \ldots)$, we have for every $\rho = 1, 2, \ldots$

$$P_\rho N \subset N P_\rho,$$

$$\left\| \frac{1}{\rho} N P_\rho x \right\| \leq \| P_\rho x \| \qquad \text{for every } x \in R,$$

and P_ρ is commutative to every projection operator which is commuta-

tive to N. Since $M_1 \subset M_2 \subset \cdots$, putting

$$P = \bigcup_{\rho=1}^{\infty} P_\rho ,$$

we have obviously $P_\rho \uparrow_{\rho=1}^{\infty} P$. Furthermore we have $PN \subset NP$. In fact, for each $\alpha \in D$ we have

$$P\alpha = \lim_{\rho\to\infty} P_\rho \alpha ,$$

$$\lim_{\rho\to\infty} NP_\rho \alpha = \lim_{\rho\to\infty} P_\rho N\alpha = PN\alpha ,$$

and hence $P\alpha \in D$, $NP\alpha = PN\alpha$, because N is closed by assumption. For such P we see by Theorems 31.2 and 31.3 that NP is a closed normal operator. Furthermore NP is regular. Because, putting

$$Q_\rho = P_\rho + (1 - P) \qquad (\rho = 1, 2, \ldots)$$

we have obviously $Q_\rho \uparrow_{\rho=1}^{\infty} 1$ and for every $\rho = 1, 2, \ldots$

$$NPQ_\rho = NP_\rho ,$$

$$\| NP_\rho x \| \leqq \rho \| P_\rho x \| \leqq \rho \| x \| \qquad \text{for every } x \in R .$$

Therefore NP is regular by definition.

For such P, we also have obviously

$$(1 - P)N \subset N(1 - P).$$

For a projection operator Q, if $QN \subset NQ$ and NQ is bounded, then we can find ρ for which

$$\| NQx \| \leqq \rho \| x \| \qquad \text{for every } x \in R ,$$

because NQ is closed by Theorem 31.3. Thus we see easily by Theorem 31.2 that for every ν, $\mu = 0, 1, 2, \ldots$ we have

$$\| (\tfrac{1}{\rho} N)^\nu (\tfrac{1}{\rho} \bar{N})^\mu Qx \| \leqq \| Qx \| \qquad \text{for every } x \in R ,$$

and hence $QR \subset M_\rho$ by Theorem 31.5. Therefore $Q \leqq P$, and hence $(1 - P)Q = 0$. Thus N is singular in $1 - P$ by definition.

For a projection operator Q, if $QN \subset NQ$ and NQ is regular, then, since NQ is closed by Theorem 31.3, we can find by Theorem 32.4 a sequence of projection operators $Q_\nu \uparrow_{\nu=1}^{\infty} 1$ such that

$$Q_\nu NQ \subset NQQ_\nu ,$$

NQQ_ν is bounded for every $\nu = 1, 2, \ldots$, and $Q_\nu R$ includes the zero manifold of NQ. Since $(1 - Q)R$ is obviously included in the zero manifold of NQ, we obtain then $1 - Q \leqq Q_\nu$ for every $\nu = 1, 2, \ldots$, and hence $Q_\nu Q = QQ_\nu$ for every $\nu = 1, 2, \ldots$. Thus we have

$$Q_\nu Q N \subset Q_\nu N Q \subset N Q Q_\nu = N Q_\nu Q .$$

Since $N Q_\nu Q$ is closed and bounded, we can find ρ_ν $(\nu = 1, 2, \ldots)$ such that $\| N Q_\nu Q x \| \leqq \rho_\nu \| x \|$ for every $x \in R$, and hence

$$\| (\frac{1}{\rho_\nu} N)^\mu (\frac{1}{\rho_\nu} \bar{N})^\sigma Q_\nu Q x \| \leqq \| x \| \qquad (\mu, \sigma = 0, 1, 2, \ldots).$$

Therefore we have $Q_\nu Q R \subset M_{\rho_\nu}$ by Theorem 31.5, and consequently

$$Q_\nu Q \leqq P_{\rho_\nu} \leqq P \qquad (\nu = 1, 2, \ldots).$$

This relation yields $Q \leqq P$, because $Q_\nu \uparrow_{\nu=1}^{\infty} 1$.

For a projection operator Q, if N is singular in Q, then we have $Q N \subset N Q$ by definition, and hence $Q P_\rho = P_\rho Q$ for every $\rho = 1, 2, \ldots$. Since $P_\rho N \subset N P_\rho$, and $N P_\rho$ is bounded for every $\rho = 1, 2, \ldots$, we obtain furthermore $Q P_\rho = 0$ for every $\rho = 1, 2, \ldots$. This relation yields $Q P = 0$, because $P_\rho \uparrow_{\rho=1}^{\infty} P$. Thus we have $Q \leqq 1 - P$.

Therefore we can now state

Theorem 32.6. *If a normal operator N is closed, then there exists uniquely a projection operator P such that $PN \subset NP$, NP is regular, and N is singular in $1 - P$.*

The regular closed normal operator NP in this Theorem 32.6 is called the regular part of a closed normal operator N.

§33 Properties of Symmetric Operators

Let S be a symmetric operator on R. It is evident by definition that S is a normal operator with domain R and S is the conjugate operator of S itself. By virtue of Theorem 10.9, S is bounded, and hence the norm $\| S \|$ of S is defined as

$$\| S \| = \sup_{\| x \| \leqq 1} \| S x \| .$$

Theorem 33.1. *For a symmetric operator S on R, we have*

$$\| S \| = \sup_{\| x \| \leqq 1} |(S x, x)| .$$

Proof. Since we have by §10(2) and §4(2) for every $x \in R$

$$|(S x, x)| \leqq \| S x \| \| x \| \leqq \| S \| \| x \|^2,$$

putting $\gamma = \sup_{\| x \| \leqq 1} |(S x, x)|$, we obtain $\gamma \leqq \| S \|$. On the other hand we have

$$|(S x, x)| \leqq \gamma \|x\|^2 \qquad \text{for every } x \in R,$$

and hence for every x, $y \in R$ and complex number λ

$$|(S(\lambda x + y), \lambda x + y)| \leqq \gamma \|\lambda x + y\|^2,$$
$$|(S(\lambda x - y), \lambda x - y)| \leqq \gamma \|\lambda x - y\|^2.$$

From this relation we conclude by the formula §4(1)

$$4|\mathcal{R}\lambda(S x, y)| \leqq 2\gamma(|\lambda|^2 \|x\|^2 + \|y\|^2).$$

Thus for every $x \neq 0$, putting

$$y = S x, \qquad \lambda = \frac{\|S x\|}{\|x\|},$$

we obtain

$$\frac{\|S x\|^3}{\|x\|} \leqq \gamma \|S x\|^2,$$

and hence $\|S x\| \leqq \gamma \|x\|$ for every $x \in R$. This relation yields obviously $\|S\| \leqq \gamma$.

For a symmetric operator S on R, $(S x, x)$ is real for every $x \in R$. In fact we have for every $x \in R$

$$(S x, x) = (x, S x) = \overline{(S x, x)}.$$

From Theorem 33.1 we conclude immediately

Theorem 33.2. *For a symmetric operator S on R, putting*

$$\alpha = \inf_{\|x\| \geqq 1} (S x, x), \qquad \beta = \sup_{\|x\| \leqq 1} (S x, x),$$

we have $\|S\| = \operatorname{Max}\{|\alpha|, |\beta|\}$.

For two symmetric operators S and T on R, we define $S \geqq T$ or $T \leqq S$ to mean

$$(S x, x) \geqq (T x, x) \qquad \text{for every } x \in R.$$

A symmetric operator S on R is said to be *positive*, if $S \geqq 0$.

Theorem 33.3. *If a symmetric operator S on R is positive, then we have*

$$|(S x, y)|^2 \leqq (S x, x)(S y, y) \qquad \text{for every } x, y \in R.$$

Proof. Since S is positive by assumption, we have for every x, $y \in R$ and complex number λ

$$(S(\lambda x - y), \lambda x - y) \geqq 0,$$

that is,

$$|\lambda|^2 (S x, x) - 2\mathcal{R}\lambda(S x, y) + (S y, y) \geqq 0.$$

We need only consider the case where $(S x, y) \neq 0$. In this case,

putting

$$\lambda = \rho \frac{\overline{(S x, y)}}{|(S x, y)|}$$

for an arbitrary real number ρ, we obtain

$$\rho^2 (S x, x) - 2\rho |(S x, y)| + (S y, y) \geqq 0.$$

If $(S x, x) = 0$, then we have

$$|(S x, y)| \leqq \frac{1}{2\rho} (S y, y) \qquad \text{for every } \rho > 0,$$

and hence $(S x, y) = 0$, contradicting the assumption. Thus we have $(S x, x) \neq 0$, and hence, putting

$$\rho = \frac{|(S x, y)|}{(S x, x)}$$

we obtain

$$(S y, y) \geqq \frac{|(S x, y)|^2}{(S x, x)}$$

that is,

$$(S x, x)(S y, y) \geqq |(S x, y)|^2.$$

Theorem 33.4. Let P be a projection operator which is commutative to a symmetric operator S on R. If there is an element $a \in R$ such that

$$(S P a, a) = \alpha \|P a\|^2, \quad \alpha = \inf_{\|P x\| \geqq 1} (S P x, x),$$

then $S P a = \alpha P a$. If there is an element $b \in R$ such that

$$(S P b, b) = \beta \|P b\|^2, \quad \beta = \sup_{\|P x\| \leqq 1} (S P x, x),$$

then $S P b = \beta P b$.

Proof. Since we have for every $x \in R$

$$((S - \alpha) P x, x) = (S P x, x) - \alpha \|P x\|^2 \geqq 0$$

and $S - \alpha$ is obviously a symmetric operator on R, we obtain by Theorem 33.3 for every $x \in R$

$$|((S - \alpha) P a, x)|^2 \leqq ((S - \alpha) P a, a)((S - \alpha) P x, x).$$

Since we have by assumption

$$((S - \alpha) P a, a) = (S P a, a) - \alpha \|P a\|^2 = 0,$$

we conclude hence

$$((S - \alpha) P a, x) = 0 \qquad \text{for every } x \in R,$$

and consequently $(S - \alpha) P a = 0$, that is, $S P a = \alpha P a$. We

also can prove likewise the other assertion.

Theorem 33.5. If a projection operator P is commutative to a symmetric operator S on R, and $Px \neq 0$ implies

$$(SPx, x) \geqq 0, \qquad \sup_{\nu \geqq 1} \| S^{\nu} Px \| = +\infty,$$

then we have $(SPx, x) \geqq \| Px \|^2$ for every $x \in R$.

Proof. We suppose $(SPa, a) < \| Pa \|^2$ for an element $a \in R$. Denoting by M_0 the closed linear manifold generated by $S^{\nu} Pa$ $(\nu = 0, 1, 2, \ldots)$, we see by Theorem 10.5 that $SM_0 \subset M_0$. Thus, denoting by P_0 the projection operator of M_0, we have $P_0 S P_0 = SP_0$, and hence by §10(10)

$$P_0 S = (SP_0)^* = (P_0 S P_0)^* = P_0 S P_0 = SP_0.$$

Furthermore it is obvious that $P_0 \leqq P$ and $P_0 a = Pa$.

Denoting by M_{ν} the linear manifold generated by $S^{\mu} Pa$ $(\mu = 0, 1, \ldots, \nu)$, we see by Theorem 2.9 that M_{ν} is closed for every $\nu = 1, 2, \ldots$. Denoting by P_{ν} the projection operator of M_{ν} $(\nu = 1, 2, \ldots)$, we have obviously $P_{\nu} \uparrow \sum_{\nu=1}^{\infty} P_0$.

Putting $S_{\nu} = P_{\nu} S P_{\nu}$ $(\nu = 1, 2, \ldots)$, we obtain a symmetric operator S_{ν} on R, because we have by §10(10)

$$S_{\nu}^* = (P_{\nu} S P_{\nu})^* = P_{\nu} S P_{\nu} = S_{\nu}.$$

By virtue of Theorem 31.5, denoting by L_{ν} the totality of those elements $x \in R$ for which

$$\sup_{\mu \geqq 1} \| S_{\nu}^{\mu} x \| < +\infty,$$

we obtain a closed linear manifold L_{ν}, and denoting by Q_{ν} the projection operator of L_{ν}, we have

$$S_{\nu} Q_{\nu} = Q_{\nu} S_{\nu},$$

$$\| S_{\nu}^{\mu} Q_{\nu} x \| \leqq \| x \| \qquad \text{for } x \in R, \mu = 1, 2, \ldots,$$

and Q_{ν} is commutative to every projection operator which is commutative to S_{ν}. Since $S_{\nu}(1 - P_{\nu}) = 0$, we have obviously

$$1 - P_{\nu} \leqq Q_{\nu}, \text{ that is, } 1 - Q_{\nu} \leqq P_{\nu}.$$

Putting

$$\alpha = \inf_{\| (1 - Q_{\nu}) x \| \geqq 1} (S_{\nu}(1 - Q_{\nu}) x, x),$$

we have $\alpha \geqq 0$, because we have by assumption for every $x \in R$

$$(S_{\nu}(1 - Q_{\nu}) x, x) = (SP(1 - Q_{\nu}) x, (1 - Q_{\nu}) x) \geqq 0.$$

We can find obviously a sequence $x_\mu \in R$ $(\mu = 1, 2, \ldots)$ such that

$$\| (1 - Q_\nu) x_\mu \| = 1 \qquad (\mu = 1, 2, \ldots),$$

$$\lim_{\mu \to \infty} (S_\nu (1 - Q_\nu) x_\mu, x_\mu) = \alpha .$$

Recalling Theorem 9.11, we can then find further a subsequence

$$(1 - Q_\nu) x_{\mu_\rho} \qquad (\rho = 1, 2, \ldots)$$

which is weakly convergent. Since $P_\nu R$ is a linear manifold generated by $S^\mu P_0 a$ $(\mu = 0, 1, 2, \ldots, \nu)$, and $(1 - Q_\nu) R \subset P_\nu R$, we see by Theorem 9.13 that this subsequence $(1 - Q_\nu) x_{\mu_\rho}$ $(\rho = 1, 2, \ldots)$ is convergent, and putting

$$x_0 = \lim_{\rho \to \infty} (1 - Q_\nu) x_{\mu_\rho},$$

we have

$$(1 - Q_\nu) x_0 = x_0, \qquad \| x_0 \| = 1,$$

$$(S_\nu (1 - Q_\nu) x_0, x_0) = \lim_{\rho \to \infty} (S_\nu (1 - Q_\nu) x_{\mu_\rho}, x_{\mu_\rho}) = \alpha .$$

Thus we have by Theorem 33.4

$$S_\nu (1 - Q_\nu) x_0 = \alpha (1 - Q_\nu) x_0 .$$

If $\alpha < 1$, then we have obviously

$$\sup_{\mu \geq 1} \| S_\nu^\mu (1 - Q_\nu) x_0 \| = \sup_{\mu \geq 1} \alpha^\mu < + \infty ,$$

and hence $(1 - Q_\nu) x_0 \in L_\nu$. Consequently we have

$$x_0 = (1 - Q_\nu) x_0 = Q_\nu (1 - Q_\nu) x_0 = 0,$$

contradicting $\| x_0 \| = 1$. Thus we have $\alpha \geqq 1$, and hence

$$(S_\nu (1 - Q_\nu) x, x) \geqq \| (1 - Q_\nu) x \| \quad \text{for every } x \in R .$$

Since $\| (1 - Q_\nu) a \| \leqq \| a \|$ $(\nu = 1, 2, \ldots)$, we can find by Theorem 9.11 a subsequence ν_ρ $(\rho = 1, 2, \ldots)$ such that $(1 - Q_{\nu_\rho}) a$ $(\rho = 1, 2, \ldots)$ is weakly convergent, and putting

$$f = \text{w-}\lim_{\rho \to \infty} (1 - Q_{\nu_\rho}) a,$$

we have $P_0 f = f$ and by Theorem 9.2

$$\| f \| \leqq \varlimsup_{\rho \to \infty} \| (1 - Q_{\nu_\rho}) a \| \leqq \varlimsup_{\rho \to \infty} \| (1 - Q_{\nu_\rho}) a \| \leqq \| P_0 a \|,$$

because $(1 - Q_\nu) \leqq P_\nu \leqq P_0$ and hence $(1 - Q_{\nu_\rho}) = (1 - Q_{\nu_\rho}) P_0$.

If $f = P_0 a$, then we have hence by Theorem 9.5

$$P_0 a = \lim_{\rho \to \infty} (1 - Q_{\nu_\rho}) a,$$

and consequently

$$(S P a, a) = (S P_0 a, a) = \lim_{\rho \to \infty} (S_{\nu_\rho} (1 - Q_{\nu_\rho}) P_0 a, P_0 a)$$

$$\geqq \lim_{\rho \to \infty} \| (1 - Q_{\nu_\rho}) P_0 a \|^2 = \| P_0 a \|^2 = \| P a \|^2,$$

contradicting the assumption $(S P a, a) < \| P a \|^2$. Therefore we have $f \neq P_0 a$. On the other hand we have

$$P_0 a - f = \text{w-}\lim_{\rho \to \infty} \{ P_0 a - P_0 (1 - Q_{\nu_\rho}) a \} = \text{w-}\lim_{\rho \to \infty} P_0 Q_{\nu_\rho} a ,$$

and it is obvious that

$$\lim_{\rho\to\infty} S_{\nu_\rho}{}^{\mu} = S^{\mu} P_0 \qquad \text{for every } \mu = 1, 2, \dots .$$

Thus we have for every $\mu = 1, 2, \dots$ by Theorem 9.4

$$\begin{aligned} \| S^{\mu}(P_0 a - \ell)\|^2 &= (S^{\mu} P_0 (P_0 a - \ell),\ S^{\mu} P_0 (P_0 a - \ell)) \\ &= (P_0 a - \ell,\ S^{2\mu} P_0 (P_0 a - \ell)) \\ &= \lim_{\rho\to\infty} (P_0 Q_{\nu_\rho} a,\ S_{\nu_\rho}^{2\mu} (P_0 a - \ell)) \\ &= \lim_{\rho\to\infty} (S_{\nu_\rho}^{2\mu} P_0 Q_{\nu_\rho} a,\ P_0 a - \ell) \\ &\leqq \lim_{\rho\to\infty} \| S_{\nu_\rho}{}^{2\mu} Q_{\nu_\rho} P_0 a \| \, \| P_0 a - \ell \| \\ &\leqq \| P_0 a \| \, \| P_0 a - \ell \| \end{aligned}$$

and hence

$$\sup_{\mu \geqq 1} \| S^{\mu}(P_0 a - \ell)\| < +\infty$$

but $P(P_0 a - \ell) = P_0 a - \ell \neq 0$. Therefore we established our assertion.

§34 Spectralisation of Normal Operators

Let $\mathfrak{R}$ be a closed ring of projection operators containing 1 in a Hilbert space R. For a dilatator T with domain D by $\mathfrak{R}$, we see by Theorem 22.4 that the adjoint operator T^* of T is a dilatator with the same domain D and we have by the formula §25(17)

$$\| T^* x \| = \| T x \| \qquad \text{for every } x \in D.$$

Therefore T is a closed normal operator and T^* is the conjugate operator of T. Furthermore T is regular by Theorem 24.4. Conversely if a normal operator N is closed and regular, then we can find a closed ring of projection operators by which N is a dilatator, as we shall prove in the sequel.

Let N be a regular closed normal operator with domain D, and $\bar{N}$ the conjugate operator of N. For every complex number ξ and positive number ε, putting

$$N_{\xi,\varepsilon} = \frac{1}{\varepsilon}(N - \xi),$$

we obtain obviously a regular closed normal operator $N_{\xi,\varepsilon}$ with the same domain D and the conjugate operator $\bar{N}_{\xi,\varepsilon}$ of $N_{\xi,\varepsilon}$ as

$$\bar{N}_{\xi,\varepsilon} = \frac{1}{\varepsilon}(\bar{N} - \bar{\xi}).$$

By virtue of Theorem 31.5, denoting by $M_{\xi,\varepsilon}$ the totality of those elements $x \in D$ for which

$$N^{\nu}\bar{N}^{\mu}x \in D \qquad \text{for every } \nu,\ \mu = 0, 1, 2, \ldots,$$

$$\sup_{\nu \geq 1} \| N_{\xi,\varepsilon}{}^{\nu}\, \bar{N}_{\xi,\varepsilon}{}^{\mu}\, x \| < +\infty \qquad \text{for every } \mu = 1, 2, \ldots,$$

we obtain a closed linear manifold $M_{\xi,\varepsilon}$. Furthermore, denoting by $P_{\xi,\varepsilon}$ the projection operator of $M_{\xi,\varepsilon}$, we see that $P_{\xi,\varepsilon}$ is commutative to N, and every projection operator, which is commutative to N, is commutative to $P_{\xi,\varepsilon}$. Therefore we have

$$P_{\xi,\varepsilon}\, P_{\xi',\varepsilon'} = P_{\xi',\varepsilon'}\, P_{\xi,\varepsilon}$$

for every complex numbers ξ, ξ' and positive numbers ε, ε'.

Putting

$$\mathfrak{E}_{\xi,\varepsilon} = \{ \zeta : |\zeta - \xi| \leqq \varepsilon \},$$

we have

(1) $\mathfrak{E}_{\xi,\varepsilon} \subset \mathfrak{E}_{\xi',\varepsilon'}$ implies $P_{\xi,\varepsilon} \leqq P_{\xi',\varepsilon'}$.

Because for every $x \in M_{\xi,\varepsilon}$ we have by Theorem 31.5

$$Nx \in M_{\xi,\varepsilon}, \quad \bar{N}x \in M_{\xi,\varepsilon}, \quad \| N_{\xi,\varepsilon}\, x \| \leqq \| x \|.$$

Thus, if $\mathfrak{E}_{\xi,\varepsilon} \subset \mathfrak{E}_{\xi',\varepsilon'}$, then $|\xi' - \xi| \leqq \varepsilon' - \varepsilon$, and hence

$$\| N_{\xi',\varepsilon'}\, x \| = \| \frac{\varepsilon}{\varepsilon'} N_{\xi,\varepsilon} x + \frac{\xi' - \xi}{\varepsilon'} x \|$$

$$\leqq \frac{\varepsilon}{\varepsilon'} \| N_{\xi,\varepsilon}\, x \| + \frac{|\xi' - \xi|}{\varepsilon'} \| x \| \leqq \| x \|.$$

Therefore we have $M_{\xi,\varepsilon} \subset M_{\xi',\varepsilon'}$ if $\mathfrak{E}_{\xi,\varepsilon} \subset \mathfrak{E}_{\xi',\varepsilon'}$.

(2) $\mathfrak{E}_{\xi,\varepsilon}\, \mathfrak{E}_{\xi',\varepsilon'} = 0$ implies $P_{\xi,\varepsilon}\, P_{\xi',\varepsilon'} = 0$.

Because for every $x \in M_{\xi,\varepsilon} M_{\xi',\varepsilon'}$ we have

$$\| N_{\xi,\varepsilon}\, x \| \leqq \| x \|, \qquad \| N_{\xi',\varepsilon'}\, x \| \leqq \| x \|,$$

and hence

$$\| (\xi' - \xi) x \| \leqq \| (N - \xi) x \| + \| (N - \xi') x \| \leqq (\varepsilon + \varepsilon') \| x \|.$$

If $\mathfrak{E}_{\xi,\varepsilon}\, \mathfrak{E}_{\xi',\varepsilon'} = 0$, then we have $|\xi' - \xi| > \varepsilon + \varepsilon'$, and hence $x = 0$. Thus we have $P_{\xi,\varepsilon}\, P_{\xi',\varepsilon'} = 0$, if $\mathfrak{E}_{\xi,\varepsilon}\, \mathfrak{E}_{\xi',\varepsilon'} = 0$.

(3) $$P_{\xi,\varepsilon} = \bigcap_{\delta > \varepsilon} P_{\xi,\delta}.$$

Because for every $x \in \prod_{\delta > \varepsilon} M_{\xi,\delta}$ we have

$$\| N_{\xi,\varepsilon}{}^{\nu}\, \bar{N}_{\xi,\varepsilon}{}^{\mu}\, x \| \leqq \left(\frac{\delta}{\varepsilon}\right)^{\nu+\mu} \| x \|$$

for every $\delta > \varepsilon$ and ν, $\mu = 0, 1, 2, \ldots$, and hence

$$\| N_{\xi,\varepsilon}{}^{\nu}\, \bar{N}_{\xi,\varepsilon}{}^{\mu}\, x \| \leqq \| x \| \qquad \text{for ever } \nu,\ \mu = 0, 1, \ldots.$$

Thus we conclude by Theorem 31.5 $x \in M_{\xi,\varepsilon}$, and hence by (1)

$$M_{\xi,\varepsilon} = \prod_{\delta > \varepsilon} M_{\xi,\delta}.$$

(4) $\mathfrak{E}_{0,\varepsilon} \subset \sum_{\nu=1}^{\kappa} \mathfrak{E}_{\xi_\nu,\varepsilon'}$ implies $P_{0,\varepsilon} \leqq \bigcup_{\nu=1}^{\kappa} P_{\xi_\nu,\varepsilon'}$.

In fact, for an arbitrary $\delta > 0$, we have for every complex number ξ

$$P_{\xi,\frac{1}{2}\delta}\,(P_{0,\varepsilon} - P_{0,\varepsilon}\overset{\varkappa}{\underset{\nu=1}{\cup}} P_{\xi_\nu,\varepsilon'+\delta}) = 0,$$

because, if $\mathfrak{E}_{0,\varepsilon}\,\mathfrak{E}_{\xi,\frac{1}{2}\delta} = 0$, then $P_{0,\varepsilon}\,P_{\xi,\frac{1}{2}\delta} = 0$ by (2), and if $\mathfrak{E}_{0,\varepsilon}\,\mathfrak{E}_{\xi,\frac{1}{2}\delta} \neq 0$, then we can find ν such that

$$\mathfrak{E}_{\xi,\frac{1}{2}\delta}\,\mathfrak{E}_{\xi_\nu,\varepsilon'} \neq 0,$$

and hence $\mathfrak{E}_{\xi,\frac{1}{2}\delta} \subset \mathfrak{E}_{\xi_\nu,\varepsilon'+\delta}$, which implies by (1)

$$P_{\xi,\frac{1}{2}\delta} \leqq P_{\xi_\nu,\varepsilon'+\delta}.$$

Thus, if

$$P_{0,\varepsilon} - P_{0,\varepsilon}\overset{\varkappa}{\underset{\nu=1}{\cup}} P_{\xi_\nu,\varepsilon'+\delta} \neq 0,$$

then we can find two positive numbers γ, γ' and a complex number ξ' such that

$$\gamma < \gamma' < \sqrt{2}\,\gamma,$$

$$P_{\xi',\gamma'}\,(P_{0,\varepsilon} - P_{0,\varepsilon}\overset{\varkappa}{\underset{\nu=1}{\cup}} P_{\xi_\nu,\varepsilon'+\delta}) \neq 0,$$

$$P_{\xi,\gamma}\,(P_{0,\varepsilon} - P_{0,\varepsilon}\overset{\varkappa}{\underset{\nu=1}{\cup}} P_{\xi_\nu,\varepsilon'+\delta}) = 0 \qquad \text{for every } \xi.$$

Putting $Q = P_{\xi',\gamma'}\,(P_{0,\varepsilon} - P_{0,\varepsilon}\overset{\varkappa}{\underset{\nu=1}{\cup}} P_{\xi_\nu,\varepsilon'+\delta})$, we see easily that $QN \subset NQ$, NQ is bounded, and

$$\left\| \frac{1}{\gamma'}\,(N - \xi')\,Q\,x \right\| \leqq \|Q\,x\| \qquad \text{for every } x \in R.$$

Furthermore, if $Q\,x \neq 0$, then we have

$$\sup_{\nu \geqq 1} \|N_{\xi,\gamma}{}^{\nu}\,Q\,x\| = +\infty \qquad \text{for every } \xi,$$

because $P_{\xi,\gamma}\,Q\,x = 0$ and $\|N_{\xi,\gamma}{}^{\nu}\,\overline{N}_{\xi,\gamma}{}^{\mu}\,Q\,x\| = \|N_{\xi,\gamma}{}^{\nu+\mu}\,Q\,x\|$. Thus we have

$$\sup_{\nu \geqq 1} \|(N_{\xi,\gamma}\,\overline{N}_{\xi,\gamma})^{\nu}\,Q\,x\| = +\infty \qquad \text{for every } \xi.$$

Since $N_{\xi,\gamma}\,\overline{N}_{\xi,\gamma}\,Q$ is obviously a symmetric operator on R and

$$(N_{\xi,\gamma}\,\overline{N}_{\xi,\gamma}\,Q\,x,\,x) = \|N_{\xi,\gamma}\,Q\,x\|^2 \geqq 0$$

for every $x \in R$, we obtain by Theorem 33.5

$$(N_{\xi,\gamma}\,\overline{N}_{\xi,\gamma}\,Q\,x,\,x) \geqq \|Q\,x\|^2 \qquad \text{for every } x \in R,$$

and hence

$$\|(N - \xi)\,Q\,x\|^2 \geqq \gamma^2\,\|Q\,x\|^2 \qquad \text{for every } \xi \text{ and } x \in R.$$

Thus, putting $\xi = \xi' \pm \gamma$, $\xi' \pm i\gamma$, we have for every $x \in R$

$$\|(N - \xi' \pm \gamma)\,Q\,x\|^2 \geqq \gamma^2\,\|Q\,x\|^2,$$

$$\|(N - \xi' \pm i\gamma)\,Q\,x\|^2 \geqq \gamma^2\,\|Q\,x\|^2,$$

and hence

$$\|(N - \xi')\,Q\,x\|^2 \geqq \gamma\,|((N - \xi' + \overline{N} - \overline{\xi'})\,Q\,x,\,x)|,$$

$$\| (N - \xi') Q x \|^2 \geqq \gamma \, | ((iN - i\xi' - i\overline{N} + i\overline{\xi'}) Q x, x) | .$$

Putting $\alpha = \sup_{\|x\| \leqq 1} \| (N - \xi') Q x \|$, we obtain therefore

$$\alpha^2 \| x \|^2 \geqq \gamma \, | ((N - \xi' + \overline{N} - \overline{\xi'}) Q x, x) | ,$$

$$\alpha^2 \| x \|^2 \geqq \gamma \, | ((iN - i\xi' - i\overline{N} + i\overline{\xi'}) Q x, x) | .$$

Since both $(N - \xi' + \overline{N} - \overline{\xi'})Q$ and $(iN - i\xi' - i\overline{N} + i\overline{\xi'})Q$ are symmetric operators on R, we obtain hence by Theorem 33.1

$$\alpha^2 \| x \| \geqq \gamma \| (N - \xi' + \overline{N} - \overline{\xi'}) Q x \| ,$$

$$\alpha^2 \| x \| \geqq \gamma \| (iN - i\xi' - i\overline{N} + i\overline{\xi'}) Q x \|$$

for every $x \in R$. These relations yield

$$2 \alpha^4 \| x \|^2 \geqq 4 \gamma^2 ((N - \xi') Q x, (N - \xi') Q x)$$

$$= 4 \gamma^2 \| (N - \xi') Q x \|^2$$

for every $x \in R$, and hence $\alpha^4 \geqq 2 \gamma^2 \alpha^2$. On the other hand, since $Q \leqq P_{\xi', \gamma'}$, we have by Theorem 31.5

$$\| (N - \xi') Q x \| \leqq \gamma' \| Q x \| \qquad \text{for every } x \in R,$$

and hence $\alpha \leqq \gamma' < \sqrt{2}\, \gamma$. Therefore we conclude $\alpha = 0$. Thus we have $Q x \in M_{\xi', \gamma}$ for every $x \in R$, and hence $Q = P_{\xi', \gamma} Q$ contradicting that $Q \neq 0$ and $Q P_{\xi', \gamma} = 0$. Therefore we have

$$P_{0, \varepsilon} - P_{0, \varepsilon} \bigcup_{\nu=1}^{\kappa} P_{\xi_\nu, \varepsilon' + \delta} = 0 \qquad \text{for every } \delta > 0.$$

Recalling (3), we obtain hence

$$P_{0, \varepsilon} = P_{0, \varepsilon} \bigcap_{\delta > 0} \bigcup_{\nu=1}^{\kappa} P_{\xi_\nu, \varepsilon' + \delta} \leqq \bigcup_{\nu=1}^{\kappa} P_{\xi_\nu, \varepsilon'} .$$

The system $P_{\xi, \varepsilon}$ for all complex numbers ξ and positive numbers ε is commutative, and hence there is a closed ring of projection operators $\mathfrak{R}$ containing 1 such that $P_{\xi, \varepsilon} \in \mathfrak{R}$ for every complex number ξ and positive number α. For such $\mathfrak{R}$, we shall prove that N is a dilatator by $\mathfrak{R}$. For this purpose we need only prove that $PN \subset NP$ for every $P \in \overline{\mathfrak{R}}$ for the commutor $\overline{\mathfrak{R}}$ of $\mathfrak{R}$.

For any $\mu = 1, 2, \ldots$ and $\varepsilon > 0$ we can find a finite number of complex numbers ξ_ν $(\nu = 1, 2, \ldots, \kappa)$ such that

$$\mathfrak{E}_{0, \mu} \subset \sum_{\nu=1}^{\kappa} \mathfrak{E}_{\xi_\nu, \varepsilon} .$$

Then we have by (4)

$$P_{0, \mu} \leqq \bigcup_{\nu=1}^{\kappa} P_{\xi_\nu, \varepsilon} .$$

Putting $P_1 = P_{\xi_1, \varepsilon} P_{0, \mu}$,

$$P_\nu = \Big(\bigcup_{\rho=1}^{\nu} P_{\xi_\rho, \varepsilon} - \bigcup_{\rho=1}^{\nu-1} P_{\xi_\rho, \varepsilon} \Big) P_{0, \mu} \qquad (\nu = 2, 3, \ldots, \kappa),$$

we see easily that $P_\nu \in \mathfrak{R}$ $(\nu = 1, 2, \ldots, \kappa)$, $P_\nu P_\rho = 0$ for $\nu \neq \rho$,

$$P_{0,\mu} = \sum_{\nu=1}^{\varkappa} P_\nu, \quad P_\nu \leqq P_{\xi_\nu,\varepsilon},$$
$$P_\nu N \subset N P_\nu \qquad (\nu = 1, 2, \ldots, \varkappa).$$

Since $\| N_{\xi_\nu,\varepsilon} P_{\xi_\nu,\varepsilon} x \| \leqq \| x \|$ for every $x \in R$, we have

$$\| (N - \xi_\nu) P_\nu x \| \leqq \varepsilon \| P_\nu x \| \qquad \text{for every } x \in R.$$

For each $P \in \overline{\mathfrak{R}}$, we have hence for every $x \in R$

$$\| (N - \xi_\nu) P_\nu P x \| \leqq \varepsilon \| P_\nu P x \| \leqq \varepsilon \| P_\nu x \|,$$
$$\| P (N - \xi_\nu) P_\nu x \| \leqq \| (N - \xi_\nu) P_\nu x \| \leqq \varepsilon \| P_\nu x \|.$$

Since $P P_\nu = P_\nu P$, $P_\nu P_\rho = 0$ for $\nu \neq \rho$, $P_{0,\mu} = \sum_{\nu=1}^{\varkappa} P_\nu$, we obtain therefore

$$\| N P_{0,\mu} P x - \sum_{\nu=1}^{\varkappa} \xi_\nu P_\nu P x \|^2 = \sum_{\nu=1}^{\varkappa} \| (N - \xi_\nu) P_\nu P x \|^2$$
$$\leqq \sum_{\nu=1}^{\varkappa} \varepsilon^2 \| P_\nu x \|^2 = \varepsilon^2 \| P_{0,\mu} x \|^2,$$
$$\| P N P_{0,\mu} x - \sum_{\nu=1}^{\varkappa} \xi_\nu P_\nu P x \|^2 = \sum_{\nu=1}^{\varkappa} \| P (N - \xi_\nu) P_\nu x \|^2$$
$$\leqq \sum_{\nu=1}^{\varkappa} \varepsilon^2 \| P_\nu x \|^2 = \varepsilon^2 \| P_{0,\mu} x \|^2,$$

and consequently

$$\| N P_{0,\mu} P x - P N P_{0,\mu} x \| \leqq 2 \varepsilon \| P_{0,\mu} x \|.$$

Since $\varepsilon > 0$ may be arbitrary, we conclude therefore

$$N P_{0,\mu} P x = P N P_{0,\mu} x \qquad \text{for every } x \in R.$$

On the other hand we have $P_{0,\mu} \uparrow_{\mu=1}^{\infty} 1$. Because, since N is regular by assumption, there exists a sequence $Q_\nu \uparrow_{\nu=1}^{\infty} 1$ such that $Q_\nu N \subset N Q_\nu$ and $N Q_\nu$ is a bounded operator on R for every $\nu = 1, 2, \ldots$. For such Q_ν $(\nu = 1, 2, \ldots)$ we can find $\mu_\nu > 0$ $(\nu = 1, 2, \ldots)$ such that

$$\| N Q_\nu x \| \leqq \mu_\nu \| Q_\nu x \| \qquad \text{for every } x \in R,$$

and hence

$$\| \left(\tfrac{1}{\mu_\nu} N\right)^\rho \left(\tfrac{1}{\mu_\nu} \bar{N}\right)^\varkappa Q_\nu x \| \leqq \| Q_\nu x \| \quad \text{for } \rho, \varkappa = 0, 1, \ldots$$

Thus we have by Theorem 31.5 $Q_\nu x \in M_{0,\mu_\nu}$ for every $x \in R$, and consequently $Q_\nu \leqq P_{0,\mu_\nu}$ for every $\nu = 1, 2, \ldots$.

Therefore we have for every $x \in D$

$$\lim_{\mu \to \infty} P_{0,\mu} P x = P x,$$
$$\lim_{\mu \to \infty} N P_{0,\mu} P x = \lim_{\mu \to \infty} P N P_{0,\mu} x = \lim_{\mu \to \infty} P_{0,\mu} P N x = P N x.$$

Since N is closed, we obtain hence $P x \in D$ and $N P x = P N x$. Thus we have $P N \subset N P$ by definition.

Now we can state

Theorem 34.1. *An operator N in R is a regular closed normal operator if and only if N is a dilatator by some closed ring of pro-*

jection operators containing 1.

For a regular closed normal operator N, considering N as a dilatator, we obtain a spectrality E on the Borel class $\mathfrak{L}$ in the complex plane $\mathfrak{G}$ and we have by Theorem 29.10

$$N = \int \zeta \, E(d\zeta).$$

Conversely, if there exists a spectrality E_1 on $\mathfrak{L}$ for which

$$N = \int \zeta \, E_1(d\zeta),$$

then N is by Theorem 29.3 a dilatator by some closed ring of projection operators, and hence N is a regular closed normal operator by Theorem 34.1. Furthermore, for the spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L})$ of N, we have by Theorem 29.5 for every $\mathfrak{E} \in \mathfrak{L}$

$$E_1(\mathfrak{E}) = E_1(\{\zeta : \zeta \in \mathfrak{E}\}) = E_1(\mathfrak{E}).$$

Therefore we have

Theorem 34.2. For a regular closed normal operator N there exists uniquely a spectrality E on the Borel class $\mathfrak{L}$ in the complex plane $\mathfrak{G}$ for which

$$N = \int \zeta \, E(d\zeta).$$

This spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L})$ is called the spectrality of a regular closed normal operator N. We have further by Theorem 29.4

$$N^* = \int \bar{\zeta} \, E(d\zeta).$$

On the other hand it is evident by definition that $\bar{N} \subset N^*$ for the conjugate operator $\bar{N}$ of N. Furthermore we see by Theorem 22.4 that the domain of N^* coincides with that of N. Consequently we obtain $\bar{N} = N^*$.

Thus we can state

Theorem 34.3. For a regular closed normal operator N, the conjugate operator $\bar{N}$ of N coincides with the adjoint operator N^* of N and

$$\bar{N} = \int \bar{\zeta} \, E(d\zeta)$$

for the spectrality $E(\mathfrak{X})$ $(\mathfrak{X} \in \mathfrak{L})$ of N.

§35 Graphs of Operators

For two linear spaces R and S, the totality of pairs $\{x, y\}$ for all $x \in R$, $y \in S$ constitutes a linear space, if we define

$$\{x_1, y_1\} + \{x_2, y_2\} = \{x_1 + x_2, y_1 + y_2\},$$
$$\alpha\{x, y\} = \{\alpha x, \alpha y\}$$

for every complex number α. This linear space will be denoted by

$$\{R, S\}.$$

If both R and S are inner product spaces, then $\{R, S\}$ also is an inner product space by the inner product

$$(\{x_1, y_1\}, \{x_2, y_2\}) = (x_1, x_2) + (y_1, y_2).$$

Furthermore, if both R and S are complete, then we see easily by definition that $\{R, S\}$ also is complete and hence a Hilbert space.

Let R be a Hilbert space. Then $\{R, R\}$ also is a Hilbert space, as defined just above. For an operator T with domain D in R, the manifold in $\{R, R\}$:

$$\{\{ix, Tx\}: \quad x \in D\}$$

is called the right graph of T, and

$$\{\{Tx, ix\}: \quad x \in D\}$$

the left graph of T.

If an operator T is linear, then it is obvious by definition that both the right and reft graphs of T are linear manifolds of $\{R, R\}$. If an operator T with domain D is closed, that is, if

$$\lim_{\nu\to\infty} x_\nu = x_0, \quad x_\nu \in D \ (\nu = 1, 2, \ldots), \quad \lim_{\nu\to\infty} Tx_\nu = y$$

implies $x_0 \in D$ and $Tx_0 = y$, then both the right and left graphs of T are closed manifolds of $\{R, R\}$, because, since

$$\|\{x, y\}\|^2 = \|x\|^2 + \|y\|^2$$

for every $x, y \in R$, we have

$$\|\{ix_\nu, Tx_\nu\} - \{ix_0, y\}\|^2 = \|\{Tx_\nu, ix_\nu\} - \{y, ix_0\}\|^2$$
$$= \|x_\nu - x_0\|^2 + \|Tx_\nu - y\|^2,$$

and hence we conclude

$$\lim_{\nu\to\infty} x_\nu = x_0, \qquad \lim_{\nu\to\infty} Tx_\nu = y$$

if and only if

$$\lim_{\nu\to\infty} \{ix_\nu, Tx_\nu\} = \{ix_0, y\}$$

or if and only if

$$\lim_{\nu\to\infty} \{T x_\nu, i x_\nu\} = \{y, i x_0\}$$

in $\{R, R\}$. Thus we have

Theorem 35.1. In order that an operator T in R be closed, it is necessary and sufficient that one of the right and left graphs of T be a closed manifold of $\{R, R\}$.

For an operator T with domain D, if D is dense in R, then we obtain the adjoint operator T^* of T by the relation

$$(Tx, y) = (x, T^*y) \qquad \text{for } x \in D,\ y \in D^*,$$

denoting by D^* the domain of T^*, as defined in §22.

Theorem 35.2. For an operator T with domain D, if D is dense in R, then the left graph of the adjoint operator T^* of T is the orthogonal complement of the right graph of T, and the right graph of T^* is the orthogonal complement of the left graph of T, that is,

$$\{\{T^*x, ix\} : x \in D^*\} = \{\{ix, Tx\} : x \in D\}^\perp,$$

$$\{\{ix, T^*x\} : x \in D^*\} = \{\{Tx, ix\} : x \in D\}^\perp.$$

Proof. For every $x \in D$, $y \in D^*$, we have by definition

$$(\{ix, Tx\}, \{T^*y, iy\}) = (ix, T^*y) + (Tx, iy) = 0.$$

Conversely if

$$(\{ix, Tx\}, \{z, iy\}) = 0$$

for every $x \in D$, then we have by definition

$$(x, z) = (Tx, y) \qquad \text{for every } x \in D,$$

and hence $y \in D^*$, $T^*y = z$. Thus we have

$$\{\{T^*x, ix\} : x \in D^*\} = \{\{ix, Tx\} : x \in D\}^\perp.$$

We also obtain likewise the other assertion.

Theorem 35.3. For an operator T with domain D, if D is dense in R, then the adjoint operator T^* of T is a closed linear operator.

Proof. By virtue of Theorems 35.1 and 35.2, we see immediately that the adjoint operator T^* of T is closed. If

$$(Tx, y) = (x, T^*y), \qquad (Tx, z) = (x, T^*z)$$

for every $x \in D$, then for any complex numbers α, β we have

$$(Tx, \alpha y + \beta z) = (x, \alpha T^* y + \beta T^* z)$$

for every $x \in D$, and hence by definition

$$T^*(\alpha y + \beta z) = \alpha T^* y + \beta T^* z .$$

Therefore T^* is a linear operator.

Theorem 35.4. If an operator T is linear and closed, and its domain D is dense in R, then the domain D^* of the adjoint operator T^* of T also is dense in R, and T is the adjoint operator of T^*, that is, $T = T^{**}$.

Proof. Recalling Theorem 35.1, we see easily by assumption that the right graph of T is a closed linear manifold of $\{R, R\}$, and hence we obtain by Theorems 11.2 and 35.2

$$\{\{ix, Tx\} : x \in D\} = \{\{ix, Tx\} : x \in D\}$$
$$= \{\{T^*x, ix\} : x \in D^*\}^{\perp}.$$

If $(y, x) = 0$ for every $x \in D^*$, then we have

$$(\{0, y\}, \{T^*x, ix\}) = -i(y, x) = 0$$

for every $x \in D^*$, and hence $y = T0 = 0$. Therefore we conclude by Theorems 2.5 and 11.2 that D^* is dense in R, because D^* is by Theorem 35.3 a linear manifold of R. Thus there exists by definition the adjoint operator T^{**} of T^*, and we see by Theorem 35.2 that $T^{**} = T$.

§36 Regularity

Let N be a normal operator in a Hilbert space R. In this § we shall consider conditions in order that N be regular.

We have obviously by definition

Theorem 36.1. If a normal operator N is regular, then $\alpha N + \beta$ also is a regular normal operator for every complex numbers α, β.

For a linear operator N with domain D, if $Nx = 0$ implies $x = 0$, then $x_1 \neq x_2 \in D$ implies $Nx_1 \neq Nx_2$, and hence putting

$$T(Nx) = x \qquad \text{for every } x \in D,$$

we obtain a linear operator T. This linear operator T is called the inverse of N and denoted by N^{-1}. The domain of the inverse N^{-1} coincides with the range of N, and

$$N^{-1}y = x \quad \text{if and only if} \quad y = Nx.$$

If N is a regular closed normal operator and has the inverse, that is, $Nx = 0$ implies $x = 0$, then N is by Theorem 34.1 a dilatator by some closed ring of projection operators $\mathfrak{R}$, and hence the inverse N^{-1} of N coincides by the formulas (31) and (45) in §25 with the inverse of N as a dilatator by $\mathfrak{R}$. Since $\bar{N} = N^*$ by Theorem 34.3, we obtain further by the formula §25(43)

$$\bar{N}^{-1} = (N^{-1})^*.$$

Thus the inverse N^{-1} also is by Theorem 34.1 a regular closed normal operator with the conjugate operator $\bar{N}^{-1}$.

Therefore we can state

Theorem 36.2. *For a regular closed normal operator N, if N has the inverse N^{-1}, then N^{-1} also is a regular closed normal operator and the conjugate operator of N^{-1} is the inverse of the conjugate operator $\bar{N}$ of N, that is, $\overline{N^{-1}} = \bar{N}^{-1}$.*

Theorem 36.3. *If the ranges of a normal operator N and its conjugate operator $\bar{N}$ coincide with the whole space R, then N is regular and closed.*

Proof. For the domain D of N, if $Nx = 0$, then we have

$$(x, \bar{N}y) = (Nx, y) = 0$$

for every $y \in D$, and hence $x = 0$, because the range of $\bar{N}$ is the whole space R by assumption. Thus N has the inverse N^{-1}. We also can prove likewise that $\bar{N}$ has the inverse $\bar{N}^{-1}$. Furthermore we have for every $x, y \in R$

$$(N^{-1}x, y) = (N^{-1}x, \bar{N}\bar{N}^{-1}y) = (NN^{-1}x, \bar{N}^{-1}y)$$
$$= (x, \bar{N}^{-1}y),$$

and for every $x \in R$, $y \in D$

$$(\bar{N}^{-1}x, Ny) = (\bar{N}\bar{N}^{-1}x, y) = (x, y) = (N\bar{N}\bar{N}^{-1}N^{-1}x, y)$$
$$= (N\bar{N}^{-1}N^{-1}x, \bar{N}y) = (N\bar{N}^{-1}N^{-1}x, Ny).$$

Since the range of N is the whole space R by assumption, we obtain hence

$$\bar{N}^{-1}x = N\bar{N}^{-1}N^{-1}x \qquad \text{for every } x \in R.$$

Thus we have for every $x \in R$

$$\|\bar{N}^{-1}x\| = \|N\bar{N}^{-1}N^{-1}x\| = \|\bar{N}\bar{N}^{-1}N^{-1}x\| = \|N^{-1}x\|.$$

Therefore N^{-1} is a normal operator and $\bar{N}^{-1}$ is the conjugate operator of N^{-1}. Since the domain of N^{-1} is the whole space R, we see by Theorem 31.1 that N^{-1} is bounded closed, and hence regular by Theorem 32.3. Since N is the inverse of N^{-1}, we conclude hence by Theorem 36.2 that N is regular and closed.

A normal operator N is said to be symmetric, if N coincides with its conjugate operator $\bar{N}$, that is, $N = \bar{N}$. With this definition we have obviously by Theorem 36.3

Theorem 36.4. *If the range of a symmetric normal operator N coincides with the whole space R, then N is regular and closed.*

Theorem 36.5. *A closed normal operator N is regular, if and only if its conjugate operator coincides with its adjoint operator, i.e.*

$$\bar{N} = N^*.$$

Proof. If N is regular and closed, then we have $\bar{N} = N^*$ by Theorem 34.3. Conversely we suppose that a normal operator N is closed and $\bar{N} = N^*$. Then we have by Theorem 35.4 $N = \bar{N}^*$, and hence by Theorem 35.2 for the domain D of N

$$\{\{ix, Nx\} : x \in D\} = \{\{\bar{N}x, ix\} : x \in D\}^{\perp},$$
$$\{\{\bar{N}x, ix\} : x \in D\} = \{\{ix, Nx\} : x \in D\}^{\perp}.$$

Therefore for each $x \in R$ we can find y, z, u, $v \in D$ such that

$$\{0, ix\} = \{iy, Ny\} + \{\bar{N}z, iz\},$$
$$\{ix, 0\} = \{iu, Nu\} + \{\bar{N}v, iv\},$$

and hence

$$0 = iy + \bar{N}z, \qquad ix = Ny + iz,$$
$$ix = iu + \bar{N}v, \qquad 0 = Nu + iv.$$

This relation yields obviously

$$x = N\bar{N}z + z = \bar{N}Nu + u.$$

Thus we have for every $w \in D$

$$(Nu, Nw) + (u, w) = (\bar{N}Nu + u, w) = (N\bar{N}z + z, w)$$
$$= (\bar{N}z, \bar{N}w) + (z, w) = (Nz, Nw) + (z, w).$$

Putting $w = u - z$, we obtain hence

$$\|N(u - z)\|^2 + \|u - z\|^2 = 0,$$

and consequently $u = z$. Therefore, corresponding to every $x \in R$ we can find uniquely $z \in D$ such that

$$x = N\bar{N}z + z = \bar{N}Nz + z.$$

Thus we have

$$N\bar{N} + 1 = \bar{N}N + 1$$

and the range of $N\bar{N} + 1$ is the whole space R. Let D_0 be the domain of $N\bar{N} + 1$. For every $x, y \in D_0$ we have

$$(N\bar{N}x + x, y) = (N\bar{N}x, y) + (x, y) = (x, N\bar{N}y) + (x, y)$$
$$= (x, N\bar{N}y + y).$$

Furthermore D_0 is dense in R. Because, if

$$(x, N\bar{N}y + y) = 0 \qquad \text{for every } x \in D_0,$$

then we have

$$(N\bar{N}x + x, y) = 0 \qquad \text{for every } x \in D_0,$$

and hence $y = 0$, since the range of $N\bar{N} + 1$ is the whole space R. Thus $N\bar{N} + 1$ is a symmetric normal operator. Consequently we see by Theorem 36.4 that $N\bar{N} + 1$ is regular and closed, and hence further by Theorem 36.1 that $N\bar{N}$ is a regular closed symmetric normal operator. Accordingly we can find by definition a sequence of projection operators $P_\nu \uparrow_{\nu=1}^{\infty} 1$ such that

$$P_\nu N\bar{N} \subset N\bar{N}P_\nu,$$
$$\| N\bar{N}P_\nu x \| \leqq \nu \| P_\nu x \| \qquad (x \in R)$$

for every $\nu = 1, 2, \ldots$. Then, since $N\bar{N} = \bar{N}N$, we have for every μ, $\rho = 0, 1, 2, \ldots$

$$\| N^\mu \bar{N}^\rho P_\nu x \|^2 = (N^\mu \bar{N}^\rho P_\nu x, N^\mu \bar{N}^\rho P_\nu x)$$
$$= (\bar{N}^\mu N^\mu \bar{N}^\rho P_\nu x, \bar{N}^\rho P_\nu x)$$
$$\leqq \| (N\bar{N})^\mu \bar{N}^\rho P_\nu x \| \| \bar{N}^\rho P_\nu x \| \leqq \nu^\mu \| \bar{N}^\rho P_\nu x \|^2,$$

and hence

$$\| N^\mu \bar{N}^\rho P_\nu x \| \leqq \sqrt{\nu}^\mu \| \bar{N}^\rho P_\nu x \|$$

for every $x \in R$. Denoting by M_ν $(\nu = 1, 2, \ldots)$ the totality of those elements $x \in D$ for which

$$\sup_{\mu \geq 1} \| \left(\frac{1}{\sqrt{\nu}} N \right)^\mu \bar{N}^\rho x \| < +\infty \qquad \text{for every } \rho = 1, 2, \ldots,$$

we obtain by Theorem 31.5 a closed linear manifold M_ν. For the projection operator P_{M_ν} of M_ν we have thus $P_\nu \leqq P_{M_\nu}$ $(\nu = 1, 2, \ldots)$, and hence $P_{M_\nu} \uparrow_{\nu=1}^{\infty} 1$. On the other hand we have by Theorem 31.5 for every $\nu = 1, 2, \ldots$

$$P_{M_\nu} N \subset N P_{M_\nu}, \qquad \| N P_{M_\nu} x \| \leqq \sqrt{\nu} \| P_{M_\nu} x \|.$$

Therefore N is regular by definition.

Theorem 36.6. For two normal operators $N \subset N_1$, if N is regular and closed, then $N = N_1$.

Proof. For the conjugate operator $\overline{N}_1$ of N_1 and the domain D_1 of N_1, we have by definition

$$(N_1 x, y) = (x, \overline{N}_1 y) \qquad \text{for every } x, y \in D_1.$$

Since $N_1 \supset N$ by assumption, for the domain D of N we have naturally

$$(Nx, y) = (x, \overline{N}_1 y) \qquad \text{for } x \in D, y \in D_1,$$

and hence $\overline{N}_1 \subset N^*$. Since $\overline{N} = N^*$ by Theorem 36.5, we obtain therefore $D = D_1$, and consequently $N = N_1$.

§37 Self-adjoint Operators

An operator H in a Hilbert space R is said to be self-adjoint, if the domain of H is dense in R and H coincides with its adjoint operator H^*, that is, $H = H^*$. Recalling Theorem 35.3, we see immediately that every self-adjoint operator is a closed linear operator. Furthermore we conclude easily by Theorem 36.5

Theorem 37.1. An operator H is self-adjoint if and only if H is a regular closed symmetric normal operator.

Let H be a self-adjoint operator in R. Since H is a regular closed symmetric normal operator by Theorem 37.1, we see by Theorem 34.1 that H is a symmetric dilatator by some closed ring of projection operators $\mathfrak{R}$ containing 1. For the proper space $\mathfrak{L}_{\mathfrak{R}}$ of such $\mathfrak{R}$, we see by Theorem 25.1 that the spectrum $(H, \mathfrak{x})$ is real for every point $\mathfrak{x} \in \mathfrak{L}_{\mathfrak{R}}$. Denoting by $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{S})$ the spectrality of H, we see by the definition §23(1) that for the real axis $(-\infty, +\infty)$ in the complex plane G we have

$$\mathfrak{U}_{E(-\infty, +\infty)} = \{\mathfrak{x} : (H, \mathfrak{x}) \in (-\infty, +\infty)\}^{-\circ} = \mathfrak{L}_{\mathfrak{R}},$$

and hence $E(-\infty, +\infty) = 1$, because $(H, \mathfrak{x})$ is finite in some open dense set. Conversely, for a regular closed normal operator H and the spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{S})$ of H, if $E(-\infty, +\infty) = 1$, then we have by Theorems 29.7 and 29.10

$$H = \int \zeta \, E(d\zeta) = \int \overline{\zeta} \, E(d\zeta) = H^*,$$

because we have by 2) in §28

$$E(\{\zeta : \zeta \neq \bar{\zeta}\}) = 1 - E(-\infty, +\infty) = 0.$$

Thus we have

Theorem 37.2. A regular closed normal operator H is self-adjoint, if and only if $E(-\infty, +\infty) = 1$ for the spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{S})$ of H and the real axis $(-\infty, +\infty)$ in the complex plane $\mathfrak{G}$.

Let H be a self-adjoint operator in $\mathfrak{R}$ and $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{S})$ the spectrality of H. Putting

$$E_\lambda = E(-\infty, \lambda) \qquad (-\infty < \lambda < +\infty)$$

for the point set

$$(-\infty, \lambda) = \{\zeta : -\infty < \zeta < \lambda\}$$

in the complex plane $\mathfrak{G}$, we see easily that

1) $E_\lambda \leqq E_\rho$ for $\lambda < \rho$,

2) $\bigcup_{\lambda<\rho} E_\lambda = E_\rho$,

3) $\bigcap_{-\infty<\lambda<+\infty} E_\lambda = 0$,

4) $\bigcup_{-\infty<\lambda<+\infty} E_\lambda = 1$,

because $E(-\infty, +\infty) = 1$ by Theorem 37.2.

A system of projection operators E_λ $(-\infty < \lambda < +\infty)$ is said to be a resolution of identity, if it satisfies the conditions 1), 2), 3), and 4). With this definition we have

Theorem 37.3. For any resolution of identity E_λ $(-\infty < \lambda < +\infty)$ there exists uniquely a spectrality E on the Borel class $\mathfrak{S}$ in the complex plane $\mathfrak{G}$ such that

$$E_\lambda = E(-\infty, \lambda) \qquad \text{for } -\infty < \lambda < +\infty.$$

Proof. E_λ $(-\infty < \lambda < +\infty)$ is obviously by 1) a commutative system. Thus there exists the closed ring $\mathfrak{R}$ generated by E_λ $(-\infty < \lambda < +\infty)$, as proved in §20. For such $\mathfrak{R}$ we have $\mathfrak{R} \ni 1$ by 4). Since we have by 3) and 4)

$$\bigcup_{\nu=1}^{\infty} (E_\nu - E_{-\nu}) = \bigcup_{\nu=1}^{\infty} E_\nu - \bigcap_{\nu=1}^{\infty} E_{-\nu} = 1,$$

we see by Theorem 15.7 that, putting $A = \sum_{\nu=1}^{\infty} \cup_{E_\nu - E_{-\nu}}$, we obtain an

open dense set A in the proper space $\mathfrak{L}_{\mathfrak{R}}$ of $\mathfrak{R}$. For any point $\mathfrak{f} \in A$, we can find obviously ν for which $\mathfrak{f} \ni E_{\nu} - E_{-\nu}$. For such ν, if

$$-\nu = \lambda_0 < \lambda_1 < \cdots < \lambda_{\kappa} = \nu,$$
$$\lambda_{\mu} - \lambda_{\mu-1} \leqq \varepsilon \qquad (\mu = 1, 2, \ldots, \kappa),$$

then we have by 1)

$$E_{\nu} - E_{-\nu} = \sum_{\mu=1}^{\kappa} (E_{\lambda_{\mu}} - E_{\lambda_{\mu-1}}).$$

Thus there exists by Theorem 15.3 uniquely μ for which

$$E_{\lambda_{\mu}} - E_{\lambda_{\mu-1}} \in \mathfrak{f},$$
$$E_{\lambda_{\rho}} - E_{\lambda_{\rho-1}} \bar{\in} \mathfrak{f} \qquad \text{for } \rho \neq \mu,$$

because $(E_{\lambda_{\rho}} - E_{\lambda_{\rho-1}})(E_{\lambda_{\mu}} - E_{\lambda_{\mu-1}}) = 0$ for $\rho \neq \mu$. Therefore, corresponding to every $\mathfrak{f} \in A$, we can find uniquely a real number $\omega(\mathfrak{f})$ such that

$$E_{\omega(\mathfrak{f})+\varepsilon} - E_{\omega(\mathfrak{f})-\varepsilon} \in \mathfrak{f} \qquad \text{for every } \varepsilon > 0.$$

This real valued function $\omega(\mathfrak{f})$ $(\mathfrak{f} \in A)$ is continuous in A. In fact, for any $\mathfrak{f} \in A$ and $\varepsilon > 0$,

$$E_{\omega(\mathfrak{f})+\varepsilon} - E_{\omega(\mathfrak{f})-\varepsilon} \in \mathfrak{y}$$

implies

$$(E_{\omega(\mathfrak{f})+\varepsilon} - E_{\omega(\mathfrak{f})-\varepsilon})(E_{\omega(\mathfrak{y})+\varepsilon} - E_{\omega(\mathfrak{y})-\varepsilon}) \neq 0,$$

and hence $|\omega(\mathfrak{f}) - \omega(\mathfrak{y})| \leqq 2\varepsilon$.

Furthermore, putting

$$\omega(\mathfrak{f}) = \infty \qquad \text{for } \mathfrak{f} \bar{\in} A,$$

we see easily that ω is a continuous function on $\mathfrak{L}_{\mathfrak{R}}$ and

$$U_{E_{\lambda-\varepsilon}} \subset \{\mathfrak{f} : \omega(\mathfrak{f}) < \lambda\} \subset U_{E_{\lambda}}$$

for $-\infty < \lambda < +\infty$, $\varepsilon > 0$. Thus for the spectrality $E(\mathfrak{F})$ $(\mathfrak{F} \in \mathfrak{L})$ by ω we obtain by §23(1)

$$U_{E_{\lambda-\varepsilon}} \subset U_{E(-\infty,\lambda)} \subset U_{E_{\lambda}},$$

and hence $E_{\lambda-\varepsilon} \leqq E(-\infty, \lambda) \leqq E_{\lambda}$ for $-\infty < \lambda < +\infty$, $\varepsilon > 0$. Since $E_{\lambda} = \bigcup_{\rho<\lambda} E_{\rho}$ by 2), we conclude therefore

$$E(-\infty, \lambda) = E_{\lambda} \qquad \text{for } -\infty < \lambda < +\infty.$$

Such spectrality $E(\mathfrak{F})$ $(\mathfrak{F} \in \mathfrak{L})$ is determined uniquely. In fact, if for two spectralities $E_1(\mathfrak{F})$, $E_2(\mathfrak{F})$ $(\mathfrak{F} \in \mathfrak{L})$

$$E_1(-\infty, \lambda) = E_2(-\infty, \lambda) \qquad (-\infty < \lambda < +\infty)$$
$$E_1(-\infty, +\infty) = E_2(-\infty, +\infty) = 1,$$

then for any open set Λ° in $\mathfrak{G}$ such that $\Lambda^\circ(-\infty,+\infty) \neq 0$, we can find a sequence $\lambda_\nu < \sigma_\nu$ $(\nu = 1, 2, \ldots)$ for which

$$\Lambda^\circ(-\infty,+\infty) = \sum_{\nu=1}^{\infty} \{\zeta : \lambda_\nu \leqq \zeta < \sigma_\nu\},$$

and hence we have

$$\begin{aligned} E_1(\Lambda^\circ) &= E_1(\Lambda^\circ(-\infty,+\infty)) \\ &= \sum_{\nu=1}^{\infty} \{E_1(-\infty,\sigma_\nu) - E_1(-\infty,\lambda_\nu)\} \\ &= \sum_{\nu=1}^{\infty} \{E_2(-\infty,\sigma_\nu) - E_2(-\infty,\lambda_\nu)\} = E_2(\Lambda^\circ). \end{aligned}$$

Thus we have $E_1(\Lambda^\circ) = E_2(\Lambda^\circ)$ for every open set Λ° in $\mathfrak{G}$.

Denoting by $\mathfrak{K}$ the totality of those Borel sets $\mathfrak{E}$ for which

$$E_1(\mathfrak{E}) = E_2(\mathfrak{E}).$$

we see easily that $\mathfrak{K}$ is a totally additive set class, and $\mathfrak{K}$ contains all open sets, as proved just above. Therefore we obtain $\mathfrak{K} = \mathfrak{L}$ by definition, and hence

$$E_1(\mathfrak{E}) = E_2(\mathfrak{E}) \qquad \text{for every } \mathfrak{E} \in \mathfrak{L}.$$

A finite function φ on the real axis $(-\infty,+\infty)$ is said to be $(\mathfrak{L})$-measurable, if for any open set Λ° in $\mathfrak{G}$ we can find $\mathfrak{E} \in \mathfrak{L}$ for which

$$\{\lambda : \varphi(\lambda) \in \Lambda^\circ\} = \mathfrak{E}(-\infty,+\infty).$$

Every $(\mathfrak{L})$-measurable function φ on $(-\infty,+\infty)$ may be extended over the whole space $\mathfrak{G}$ as a $(\mathfrak{L})$-measurable function. For example, putting

$$\Psi(\zeta) = \begin{cases} 0 & \text{for } \zeta \bar{\in} (-\infty,+\infty) \\ \varphi(\zeta) & \text{for } \zeta \in (-\infty,+\infty), \end{cases}$$

we obtain a $(\mathfrak{L})$-measurable function Ψ on $\mathfrak{G}$, because $(-\infty,+\infty) \in \mathfrak{L}$.

Let E_λ $(-\infty < \lambda < +\infty)$ be a resolution of identity. There exists by Theorem 37.3 uniquely a spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L})$ such that

$$E(-\infty,\lambda) = E_\lambda \qquad (-\infty < \lambda < +\infty).$$

Since $E(-\infty,+\infty) = 1$, we may consider the spectrality E as a spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in (-\infty,+\infty)\mathfrak{L})$ on $(-\infty,+\infty)\mathfrak{L}$. Thus for $(\mathfrak{L})$-measurable functions φ on the real axis $(-\infty,+\infty)$, we can define integral of φ by $E_\lambda a$ $(-\infty < \lambda < +\infty)$ as

$$\int_\alpha^\beta \varphi(\lambda)\, d_\lambda E_\lambda a = \int_{(\alpha,\beta)} \varphi(\zeta) E(d\zeta) a,$$

$$\int_\alpha^\beta \varphi(\lambda)\, d_\lambda (E_\lambda a, b) = \int_{(\alpha,\beta)} \varphi(\zeta)\, (E(d\zeta) a, b),$$

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, d_\lambda E_\lambda a = \int_{(-\infty,+\infty)} \varphi(\zeta) E(d\zeta) a.$$

$$\int_{-\infty}^{+\infty} \varphi(\lambda) d_\lambda (E_\lambda a, b) = \int_{(-\infty,+\infty)} \varphi(\zeta)\, (E(d\zeta)a, b).$$

Thus, if φ is a finite continuous function on $(-\infty, +\infty)$, then we have

$$\int_\alpha^\beta \varphi(\lambda)\, d_\lambda E_\lambda a = \lim_{\delta \to 0} \sum_{\nu=1}^{\kappa} \varphi(\xi_\nu)\, (E_{\lambda_\nu} - E_{\lambda_{\nu-1}}) a$$

for $\alpha = \lambda_0 < \lambda_1 < \cdots < \lambda_\kappa = \beta$, $\lambda_\nu - \lambda_{\nu-1} \leqq \delta$,

$\lambda_{\nu-1} \leqq \xi_\nu \leqq \lambda_\nu \qquad (\nu = 1, 2, \ldots, \kappa)$,

and

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, d_\lambda E_\lambda a = \lim_{\nu \to \infty} \int_{-\nu}^{\nu} \varphi(\lambda)\, d_\lambda E_\lambda a.$$

We also define further integral operator by φ as

$$\int \varphi(\lambda) d_\lambda E_\lambda = \int \varphi(\zeta) E(d\zeta).$$

With this definition we have obviously by Theorems 34.2 and 37.2

Theorem 37.4. For a self-adjoint operator H there exists uniquely a resolution of identity E_λ $(-\infty < \lambda < +\infty)$ such that

$$H = \int \lambda\, d_\lambda E_\lambda.$$

Conversely we have by Theorem 29.4

Theorem 37.5. For every resolution of identity E_λ $(-\infty < \lambda < +\infty)$ the integral operator

$$\int \lambda\, d_\lambda E_\lambda$$

is a self-adjoint operator.

A self-adjoint operator H with domain D is said to be positive, if

$$(Hx, x) \geqq 0 \qquad \text{for every } x \in D.$$

With this definition we have

Theorem 37.6. For a resolution of identity E_λ $(-\infty < \lambda < +\infty)$, the integral operator

$$\int \lambda\, d_\lambda E_\lambda$$

is positive, if and only if $E_0 = 0$.

Proof. If $E_0 = 0$, then, putting

$$H = \int \lambda \, d_\lambda E_\lambda ,$$

we have by Theorem 28.1 for every element x in the domain of H

$$(Hx, x) = \int_0^{+\infty} \lambda \, d_\lambda \| E_\lambda x \|^2 \geqq 0.$$

If $E_0 \neq 0$, then we can find by 2) $\varepsilon > 0$ such that $E_{-\varepsilon} \neq 0$, and hence there is an element a in the domain of H such that

$$E_{-\varepsilon} a \neq 0.$$

For such $\varepsilon > 0$ and a, we have

$$(H E_{-\varepsilon} a, E_{-\varepsilon} a) = \int_{-\infty}^{-\varepsilon} \lambda \, d_\lambda \| E_\lambda a \|^2$$

$$\leqq -\varepsilon \int_{-\infty}^{-\varepsilon} d_\lambda \| E_\lambda a \|^2 = -\varepsilon \| E_{-\varepsilon} a \|^2 < 0.$$

Recalling Theorem 25.1, we conclude immediately

Theorem 37.7. A dilatator H by a closed ring of projection operators $\mathfrak{R}$ is a self-adjoint operator if and only if the spectrum $(H, \mathfrak{p})$ is real for every point of the proper space $\mathfrak{L}_{\mathfrak{R}}$ of $\mathfrak{R}$.

Furthermore we have by Theorem 25.2

Theorem 37.8. A dilatator H by a closed ring of projection operators $\mathfrak{R}$ is a positive self-adjoint operator if and only if we have $(H, \mathfrak{p}) \geqq 0$ for every point $\mathfrak{p}$ in the proper space $\mathfrak{L}_{\mathfrak{R}}$ of $\mathfrak{R}$.

§38 Isometric Operators

A bounded linear operator U on a Hilbert space R is said to be isometric, if denoting by A the zero manifold of U we have

$$\| U x \| = \| x \| \quad \text{for every } x \in A^{\perp}.$$

For the zero manifold A of an isometric operator U, $A^{\perp}$ is called the proper domain of U.

Let U be an isometric operator and P_U the projection operator of the proper domain of U. Then we have obviously by definition

$$U P_U x = U x, \quad \| U P_U x \| = \| P_U x \| \quad \text{for every } x \in R.$$

For an arbitrary complex number λ, since

$$\| U P_U (x + \lambda y) \|^2 = \| U P_U x \|^2 + 2 \Re \bar{\lambda} (U P_U x, U P_U y) + |\lambda|^2 \| U P_U y \|^2 ,$$

$$\| P_U (x + \lambda y) \|^2 = \| P_U x \|^2 + 2 \Re \bar{\lambda} (P_U x, P_U y) + |\lambda|^2 \| P_U y \|^2 ,$$

we obtain hence

$$\Re \bar{\lambda} (U P_U x, U P_U y) = \Re \bar{\lambda} (P_U x, P_U y).$$

Putting $\lambda = 1$ and i, we conclude thus

$$(U P_U x, U P_U y) = (P_U x, P_U y),$$

and consequently

$$(Ux, Uy) = (x, P_U y) \qquad \text{for every } x, y \in R.$$

Therefore, for the adjoint operator U^* of U we have by definition

$$U^* U y = P_U y \qquad \text{for every } y \in R.$$

Since $\| P_U y \| = \| U y \|$, we obtain hence

$$\| U^* U x \| = \| U x \| \qquad \text{for every } x \in R.$$

Let B be the range of an isometric operator U. B is obviously a linear manifold. Furthermore B is closed. In fact, if

$$\lim_{\nu \to \infty} U x_\nu = y,$$

then we have

$$\lim_{\mu, \nu \to \infty} \| P_U x_\mu - P_U x_\nu \| = \lim_{\mu, \nu \to \infty} \| U x_\mu - U x_\nu \| = 0.$$

Thus we can find $x_0 \in R$ for which $\lim_{\nu \to \infty} P_U x_\nu = x_0$, and then we have

$$U x_0 = \lim_{\nu \to \infty} U P_U x_\nu = \lim_{\nu \to \infty} U x_\nu = y.$$

Thus B is a closed linear manifold. For each $y \in B^\perp$, we have for every $x \in R$

$$(U^* y, x) = (y, Ux) = 0,$$

and hence we conclude that $U^* y = 0$ for every $y \in B^\perp$.

Therefore we can state

Theorem 38.1. *For every isometric operator U, the adjoint operator U^* of U also is isometric, and the range of U is the proper domain of U^*. For the projection operator P_U of the proper domain of U we have*

$$U^* U = P_U .$$

From this Theorem 38.1 we conclude immediately

Theorem 38.2. For an isometric operator U and its adjoint operator U^* we have

$$U U^* U = U, \qquad U^* U U^* = U^*.$$

Theorem 38.3. An isometric operator U is a normal operator if and only if the proper domain of U coincides with the range of U.

Proof. If an isometric operator U is a normal operator, then the adjoint operator U^* of U is the conjugate operator of U, and hence

$$\| U x \| = \| U^* x \| \qquad \text{for every } x \in R.$$

Thus the zero manifold of U coincides with that of U^*, and consequently the proper domain of U coincides with that of U^*. On the other hand, the proper domain of U^* coincides by Theorem 38.1 with the range of U. Therefore the range of U coincides with the proper domain of U.

Conversely, if the range of U coincides with the proper domain of U, then, denoting by D the proper domain of U, we see by Theorem 38.1 that D also is the proper domain of the adjoint operator U^* of U. Thus for the projection operator P_D of D we have

$$\| U x \| = \| U P_D x \| = \| P_D x \| = \| U^* P_D x \| = \| U^* x \|$$

for every $x \in R$. Therefore U is a normal operator and U^* is the conjugate operator of U.

An operator U on R is said to be unitary, if U is isometric and both the proper domain and the range of U coincide with the whole space R. By virtue of Theorem 38.1, we see easily that if U is unitary, then its adjoint operator U^* also is unitary and

$$U^* U = U U^* = 1.$$

Conversely, for an operator U on R, if $U^* U = U U^* = 1$, then both the domain and the range of U^* coincide with the whole space R and

$$\| U^* x \|^2 = (U U^* x, x) = \| x \|^2 \qquad \text{for every } x \in R.$$

Furthermore U^* is by Theorem 35.3 a linear operator. Thus U^* is a unitary operator. Since $U = U^{**}$, we conclude hence that U is a unitary operator. Therefore we have

Theorem 38.4. An operator U on R is unitary if and only if

$$U^* U = U U^* = 1$$

<u>for the adjoint operator</u> U^* <u>of</u> U.

Let U be a unitary operator on R. By virtue of Theorem 38.3, U is a normal operator and the adjoint operator U^* of U is the conjugate operator of U. Since U is bounded, we see by Theorem 32.3 that U is a regular closed normal operator. Thus we can find by Theorem 34.1 a closed ring of projection operators $\mathfrak{R}$ containing 1, by which U is a dilatator. For such $\mathfrak{R}$, since $U U^* = 1$, we obtain by the formulas (9) and (12) in §25

$$|(U, \mathfrak{x})|^2 = (U, \mathfrak{x})\,(U^*, \mathfrak{x}) = 1$$

for every point $\mathfrak{x}$ in the proper space $\mathfrak{L}_{\mathfrak{R}}$ of $\mathfrak{R}$. Thus putting

$$\mathfrak{E}_\lambda = \{ e^{\xi i} : \ 0 \leqq \xi < \lambda \} \quad \text{for } 0 \leqq \lambda \leqq 2\pi,$$

we have

$$(U, \mathfrak{x}) \in \mathfrak{E}_{2\pi} \quad \text{for every } \mathfrak{x} \in \mathfrak{L}_{\mathfrak{R}}.$$

Consequently we see by §23(1) that for the spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{S})$ of U we have

$$E(\mathfrak{E}_{2\pi}) = 1.$$

Therefore putting

$$E_\lambda = \begin{cases} 0 & \text{for } \lambda \leqq 0 \\ E(\mathfrak{E}_\lambda) & \text{for } 0 < \lambda \leqq 2\pi \\ 1 & \text{for } \lambda > 2\pi, \end{cases}$$

we obtain a resolution of identity E_λ $(-\infty < \lambda < +\infty)$, and we have obviously by definition

$$U x = \int_{\mathfrak{E}_{2\pi}} \zeta \, E(d\zeta) x = \int_0^{2\pi} e^{i\lambda} \, d_\lambda E_\lambda x$$

for every $x \in R$. Conversely if

$$U = \int_0^{2\pi} e^{i\lambda} \, d_\lambda E_\lambda$$

for a resolution of identity E_λ $(-\infty < \lambda < +\infty)$ subject to the condition: $E_0 = 0$, $E_{2\pi+0} = 1$, we have by Theorem 29.5

$$E(\mathfrak{E}_\lambda) = E_\lambda \quad \text{for } 0 < \lambda < 2\pi.$$

Therefore we can state

<u>Theorem</u> 38.5. <u>For a unitary operator</u> U <u>on</u> R <u>there exists uniquely a resolution of identity</u> E_λ $(-\infty < \lambda < +\infty)$ <u>such that</u>

$$U = \int_0^{2\pi} e^{i\lambda}\, d_\lambda E_\lambda ,$$

$$E_0 = 0, \qquad E_{2\pi+0} = \bigcap_{\lambda>2\pi} E_\lambda = 1.$$

Furthermore we have by Theorem 38.4

Theorem 38.6. A dilatator U by a closed ring of projection operators $\mathfrak{R}$ is a unitary operator if and only if $|(U, \mathfrak{x})| = 1$ for every point $\mathfrak{x}$ in the proper space $\mathfrak{L}_{\mathfrak{R}}$ of $\mathfrak{R}$.

§39 Canonical Form

Let T be a closed linear operator with dense domain D in a Hilbert space R. Then we see by Theorem 35.4 that the adjoint operator T^* of T also is a closed linear operator with dense domain D^* and $T = T^{**}$. Thus we have by Theorem 35.2

$$\{\{ix, Tx\} : x \in D\} = \{\{T^*x, ix\} : x \in D^*\}^\perp,$$
$$\{\{T^*x, ix\} : x \in D^*\} = \{\{ix, Tx\} : x \in D\}^\perp.$$

Therefore for any $z \in R$ we can find $x \in D$ and $y \in D^*$ such that

$$\{iz, 0\} = \{ix, Tx\} + \{T^*y, iy\},$$

that is,

$$iz = ix + T^*y, \qquad Tx + iy = 0.$$

This relation yields

$$z = T^*Tx + x.$$

Thus the range of the operator $T^*T + 1$ is the whole space R. Denoting by D_0 the domain of $T^*T + 1$, we have for every $x, y \in D_0$

$$(T^*Tx + x, y) = (x, T^*Ty + y).$$

Accordingly, if

$$(x, T^*Ty + y) = 0 \qquad \text{for every } x \in D_0,$$

then we have

$$(T^*Tx + x, y) = 0 \qquad \text{for every } x \in D_0,$$

and hence $y = 0$, because $(T^*T + 1)D_0 = R$. Therefore D_0 is dense in R. Consequently we see by Theorem 36.4 that $T^*T + 1$ is a regular closed symmetric normal operator, namely a self-adjoint operator by Theorem 37.1. Thus T^*T also is by Theorem 36.1 a self-adjoint operator with the same domain D_0. By virtue of Theo-

rem 34.1, we may consider $T^* T$ as a dilatator by a closed ring of projection operators $\mathfrak{R}$. Since

$$(T^* T x, x) = \| T x \|^2 \geqq 0 \qquad \text{for every } x \in \mathfrak{D}_0,$$

we have $T^* T \geqq 0$ by Theorem 25.2, and hence $T^* T = | T^* T |$. Thus, putting $H = \sqrt{T^* T}$, we obtain by the formula §25(27)

$$T^* T = H \circ H = H H .$$

By virtue of Theorem 24.4, denoting by $\mathfrak{D}_H$ the domain of H we can find then a sequence $\mathfrak{R} \ni P_\nu \uparrow_{\nu=1}^{\infty} 1$ such that $P_\nu R \subset \mathfrak{D}_H$ and $H P_\nu$ is bounded for every $\nu = 1, 2, \ldots$. For such P_ν $(\nu = 1, 2, \ldots)$, we have obviously for every $x \in R$

$$T^* T P_\nu x = H H P_\nu x \qquad (\nu = 1, 2, \ldots),$$

and hence

$$\| H P_\nu x \|^2 = (H H P_\nu x, P_\nu x) = (T^* T P_\nu x, P_\nu x)$$
$$= \| T P_\nu x \|^2 .$$

If $x \in \mathfrak{D}_H$, then we have

$$\lim_{\nu \to \infty} H P_\nu x = \lim_{\nu \to \infty} P_\nu H x = H x ,$$

and hence

$$\lim_{\nu, \mu \to \infty} \| T P_\nu x - T P_\mu x \| = \lim_{\nu, \mu \to \infty} \| H P_\nu x - H P_\mu x \| = 0.$$

Since T is closed and $\lim_{\nu \to \infty} P_\nu x = x$, we obtain hence $x \in \mathfrak{D}$ and

$$\lim_{\nu \to \infty} T P_\nu x = T x .$$

Thus we have $\mathfrak{D} \supset \mathfrak{D}_H$ and

$$\| T x \| = \| H x \| \qquad \text{for every } x \in \mathfrak{D}_H .$$

Thereofre, denoting by $\mathfrak{A}$ the closed linear manifold generated by the range of H: $\{ H x : x \in \mathfrak{D}_H \}$, we obtain uniquely an isometric operator U with the proper space $\mathfrak{A}$ such that

$$T x = U H x \qquad \text{for every } x \in \mathfrak{D}_H ,$$

that is, $T \supset U H$. On the other hand, $U H$ is closed. In fact, if $\lim_{\nu \to \infty} x_\nu = a$ and $\lim_{\nu \to \infty} U H x_\nu = b$, then we have

$$\lim_{\nu, \mu \to \infty} \| H x_\nu - H x_\mu \| = \lim_{\nu, \mu \to \infty} \| U H x_\nu - U H x_\mu \| = 0.$$

Since H is closed, we have hence $a \in \mathfrak{D}_H$ and

$$H a = \lim_{\nu \to \infty} H x_\nu .$$

This relation yields

$$U H a = \lim_{\nu \to \infty} U H x_\nu = b .$$

Recalling Theorem 35.1, we see hence that the right graph of $U H$

$$\{ \{ x, U H x \} : x \in \mathfrak{D}_H \}$$

is a closed linear manifold of $\{R, R\}$. Thus for any $a \in D$ we can find $x_0 \in D_H$ for which

$$\{ia, Ta\} - \{ix_0, UHx_0\} \perp \{\{ix, UHx\} : x \in D_H\}.$$

Since $D_H \supset D_0$ and $UHx = Tx$ for $x \in D_H$, we obtain hence for every $x \in D_0$

$$(\{i(a - x_0), T(a - x_0)\}, \{ix, Tx\}) = 0,$$

that is,

$$(a - x_0, x) + (T(a - x_0), Tx) = 0.$$

Thus we have

$$(a - x_0, T^*Tx + x) \geqq 0 \qquad \text{for every } x \in D_0.$$

Since $R = (T^*T + 1)D_0$, we obtain hence $a = x_0$. Therefore we conclude $D_H = D$, and consequently $T = UH$.

Conversely if $T = UH$ for a positive self-adjoint operator H and an isometric operator U whose proper domain is the closed linear manifold generated by the range of H, then we have $T^* = HU^*$. Because for each $y \in D^*$, we have for every $x \in D$

$$(Hx, U^*y) = (UHx, y) = (x, T^*y),$$

and hence $T^*y = H^*U^*y = HU^*y$. On the other hand if HU^*y has any sense, then we have for every $x \in D$

$$(Tx, y) = (Hx, U^*y) = (x, HU^*y),$$

and hence $T^*y = HU^*y$ by definition. Thus we have $T^* = HU^*$. Since the proper domain of U includes the range of H, we obtain by Theorem 38.1

$$T^*T = (HU^*)(UH) = HH,$$

and hence $H = \sqrt{T^*T}$, because H is positive.

Now we can state

Theorem 39.1. *For a closed linear operator T with dense domain we can find uniquely a positive self-adjoint operator H and an isometric operator U such that $T = UH$ and the proper domain of U is the closed linear manifold generated by the range of H.*

This UH is called the canonical form of a closed linear operator T with dense domain. Concerning the canonical form of T, we have also proved

Theorem 39.2. For the canonical form $T = UH$, we have

$$T^* = HU^*, \qquad T^*T = HH,$$

and T^*T is a positive self-adjoint operator.

Theorem 39.3. A closed linear operator T with dense domain is a regular normal operator, if and only if for the canonical form $T = UH$, U is a normal operator and $UH = HU$.

Proof. If an isometric operator U is a normal operator and $UH = HU$, then we see by Theorem 23.3 that for the spectrality $E_U(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L})$ of U and $E_H(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L})$ of H, we have

$$E_U(\mathfrak{E})E_H(\mathfrak{F}) = E_H(\mathfrak{F})E_U(\mathfrak{E}) \qquad \text{for every } \mathfrak{E}, \mathfrak{F} \in \mathfrak{L}.$$

Thus, denoting by $\mathfrak{R}$ the closed ring generated by

$$E_U(\mathfrak{E}), \quad E_H(\mathfrak{E}) \qquad (\mathfrak{E} \in \mathfrak{L}),$$

U and H may be considered as dilatators by $\mathfrak{R}$. Then we have by Theorem 25.6

$$HU = H \circ U.$$

Since $T = UH = HU$ by assumption, T also is a dilatator by $\mathfrak{R}$, and hence a regular closed normal operator by Theorem 34.1.

Conversely if T is a regular closed normal operator, then we can find by Theorem 34.1 a closed ring of projection operators $\mathfrak{R}$ by which T is a dilatator. For the canonical form $T = UH$, since we have $HH = T^*T$ by Theorem 39.2, we see easily that H also is a dilatator by $\mathfrak{R}$ and we have

$$T \circ H^{-1} = U.$$

Therefore U also is a dilatator by $\mathfrak{R}$, and hence we have by Theorem 25.6

$$T = U \circ H = H \circ U = HU.$$

Theorem 39.4. A closed linear operator T with dense domain is a regular normal operator if and only if

$$TT^* = T^*T$$

for the adjoint operator T^* of T.

Proof. If $TT^* = T^*T$, then for the canonical form UH of T and that U_1H_1 of T^*, we have by Theorem 39.2

$$H_1 H_1 = TT^* = T^*T = HH,$$

and hence we conclude easily

$$H_1 = \sqrt{H_1 H_1} = \sqrt{H H} = H.$$

Thus we have

$$\|Tx\| = \|UHx\| = \|Hx\| = \|H_1 x\|$$
$$= \|UH_1 x\| = \|T^* x\|.$$

Therefore T is a normal operator and T^* is the conjugate operator of T. Consequently T is regular by Theorem 36.5.

Conversely if T is a regular closed normal operator, then, considering T as a dilatator by some closed ring of projection operators $\mathfrak{R}$, we have by the formula §25(5)

$$T \circ T^* = T^* \circ T.$$

On the other hand we have by the formula §25(11)

$$T^* \circ T \supset T^* T$$

and both $T^* \circ T$ and $T^* T$ are self-adjoint. Thus we conclude by Theorem 36.6 $T^* \circ T = T^* T$. We also obtain likewise $T \circ T^* = TT^*$. Consequently we have $TT^* = T^* T$.

§40 Hermitean Operators

An operator H in a Hilbert space R is said to be a Hermitean operator, if H is a closed symmetric normal operator, i.e., the domain D of H is a linear manifold being dense in R,

$$(Hx, y) = (x, Hy) \qquad \text{for every } x, y \in D,$$

and H is closed.

Let H be a Hermitean operator with domain D. For each $x \in D$ we have

$$\|(H+i)x\|^2 = \|(H-i)x\|^2 = \|Hx\|^2 + \|x\|^2 \geqq \|x\|^2.$$

If $\lim_{\nu \to \infty} (H+i)x_\nu = f$, then, since

$$\|x_\nu - x_\mu\| \leqq \|(H+i)x_\nu - (H+i)x_\mu\| \qquad (\nu, \mu = 1, 2, \ldots),$$

we have $\lim_{\nu, \mu \to \infty} \|x_\nu - x_\mu\| = 0$, and hence putting $a = \lim_{\nu \to \infty} x_\nu$, we have $(H+i)a = f$, because $H + i$ also is closed. Therefore $(H+i)D$ is a closed linear manifold. We also can prove likewise that $(H-i)D$ is a closed linear manifold.

Theorem 40.1. For a Hermitean operator H with domain D,

$$((H+i)D)^\perp, \qquad ((H-i)D)^\perp, \qquad D$$

are linearly independent, i.e.,

$$u + v + a = 0,$$

$$u \in ((H+i)D)^{\perp}, \quad v \in ((H-i)D)^{\perp}, \quad a \in D$$

implies $u = v = a = 0$.

Proof. For any $u \in ((H+i)D)^{\perp}$ and $v \in ((H-i)D)^{\perp}$ we have obviously for every $x \in D$

$$((H+i)x, u) = 0, \qquad ((H-i)x, v) = 0,$$

and hence

$$(Hx, u) = (x, iu), \qquad (Hx, v) = (x, -iv).$$

Thus, if $u + v + a = 0$ for some $a \in D$, then we have for every $x \in D$

$$\begin{aligned}(x, iu - iv + Ha) &= (x, iu) + (x, -iv) + (x, Ha) \\ &= (Hx, u) + (Hx, v) + (Hx, a) \\ &= (Hx, u + v + a) = 0.\end{aligned}$$

Since D is dense in $\mathfrak{H}$, we obtain hence

$$iu - iv + Ha = 0.$$

Since $u + v + a = 0$ by assumption, we conclude further

$$2iu + (H+i)a = 0, \qquad -2iv + (H-i)a = 0.$$

On the other hand we have obviously

$$(2iu, (H+i)a) = 0, \qquad (-2iv, (H-i)a) = 0.$$

Thus we obtain by the formula §7(4)

$$\|2iu\|^2 + \|(H+i)a\|^2 = 0, \quad \|-2iv\|^2 + \|(H-i)a\|^2 = 0.$$

This relation yields $u = v = 0$, and consequently $a = 0$.

For a Hermitean operator H, the dimension of the closed linear manifold $((H+i)D)^{\perp}$ is called the positive defect of H, and the dimension of $((H-i)D)^{\perp}$ is called the negative defect of H.

Since $H + i$ is a normal operator with the conjugate operator $H - i$, we see easily by Theorems 36.2 and 36.3

Theorem 40.2. A Hermitean operator H is regular if and only if both the positive and negative defects are 0.

Theorem 40.3. For a Hermitean operator H, in order that we can find a self-adjoint operator $\widetilde{H} \supset H$, it is necessary and sufficient that the positive defect of H coincides with the negative defect

of H.

Proof. If there is a self-adjoint operator $\widetilde{H} \supset H$, then we see by Theorem 36.1 that $\widetilde{H} + i$ is a regular closed normal operator with the conjugate operator $\widetilde{H} - i$. Denoting by $\widetilde{D}$ the domain of $\widetilde{H}$, we have obvously for every $x \in \widetilde{D}$

$$\| (\widetilde{H} + i)x \|^2 = \| (\widetilde{H} - i)x \|^2 = \| \widetilde{H} x \|^2 + \| x \|^2 \geqq \| x \|^2.$$

Thus we see by Theorem 36.2 that the inverse $(\widetilde{H} + i)^{-1}$ is a bounded closed normal operator with the conjugate operator $(\widetilde{H} - i)^{-1}$, and hence both $(\widetilde{H} + i)\widetilde{D}$ and $(\widetilde{H} - i)\widetilde{D}$ coincide with the whole space $\mathcal{R}$. Furthermore, putting

$$(\widetilde{H} - i)x = U(\widetilde{H} + i)x \qquad \text{for } x \in \widetilde{D},$$

we obtain a unitary operator U. For this unitary operator U, we have obviously

$$(H - i)x = U(H + i)x \qquad \text{for every } x \in D.$$

Thus we conclude easily that the dimension of $((H + i)D)^{\perp}$ coincides with that of $((H - i)D)^{\perp}$, that is, the positive defect of H coincides with the negative defect of H.

Conversely if the positive defect of H coincides with the negative defect of H, then we can find by definition a complete orthonormal system a_λ $(\lambda \in \Lambda)$ of $((H + i)D)^{\perp}$ and b_λ $(\lambda \in \Lambda)$ of $((H - i)D)^{\perp}$ with the same density. Putting

$$\sum_{\lambda \in \Lambda} \alpha_\lambda b_\lambda + (H - i)x = U \{ \sum_{\lambda \in \Lambda} \alpha_\lambda a_\lambda + (H + i)x \}$$

for $\sum_{\lambda \in \Lambda} |\alpha_\lambda|^2 < +\infty$, $x \in D$, we see easily that U is a unitary operator. For this unitary operator U, putting

$$i(y + Uy) = \widetilde{H}(y - Uy) \qquad \text{for } y \in R,$$

we obtain an operator $\widetilde{H}$ with domain

$$\widetilde{D} = \{ y - Uy : \ y \in R \}.$$

Because, if $y = Uy$ for some $y \in R$, then putting

$$y = \sum_{\lambda \in \Lambda} \alpha_\lambda a_\lambda + (H + i)x,$$

we have

$$\sum_{\lambda \in \Lambda} \alpha_\lambda b_\lambda + (H - i)x = \sum_{\lambda \in \Lambda} \alpha_\lambda a_\lambda + (H + i)x,$$

that is,

$$\sum_{\lambda \in \Lambda} \alpha_\lambda b_\lambda - \sum_{\lambda \in \Lambda} \alpha_\lambda a_\lambda - 2ix = 0.$$

Since

$$\sum_{\lambda\in\Lambda}\alpha_\lambda\beta_\lambda\in((H-i)D)^\perp,\quad \sum_{\lambda\in\Lambda}\alpha_\lambda a_\lambda\in((H+i)D)^\perp,$$

and $x\in D$, we obtain by Theorem 40.1

$$\sum_{\lambda\in\Lambda}\alpha_\lambda a_\lambda = 0,\qquad x = 0,$$

and hence $y = 0$. If $y_1 - Vy_1 = y_2 - Vy_2$, then we have

$$(y_1 - y_2) = V(y_1 - y_2),$$

and hence $y_1 = y_2$, as proved just above.

For every $x\in D$ we have

$$(H+i)x - V(H+i)x = (H+i)x - (H-i)x = 2ix,$$

$$i\{(H+i)x + V(H+i)x\} = i\{(H+i)x + (H-i)x\} = 2iHx,$$

and hence $\widetilde{H}\supset H$. For every $y, z\in R$ we have

$$\begin{aligned}(\widetilde{H}(y - Vy), z - Vz) &= (i(y + Vy), z - Vz)\\ &= i(y,z) + i(Vy,z) - i(y,Vz) - i(Vy,Vz)\\ &= i(Vy,Vz) + i(Vy,z) - i(y,Vz) - i(y,z)\\ &= (y - Vy, i(z + Vz)) = (y - Vy, \widetilde{H}(z - Vz)),\end{aligned}$$

and hence $\widetilde{H}$ is a symmetric normal operator. Furthermore we have for every $y\in R$

$$\widetilde{H}(y - Vy) + i(y - Vy) = 2iy,$$
$$\widetilde{H}(y - Vy) - i(y - Vy) = 2iVy.$$

Thus we conclude by Theorems 36.3 and 36.1 that $\widetilde{H}$ is regular and closed. Consequently $\widetilde{H}$ is self-adjoint.

Theorem 40.4. *For a Hermitean operator H with domain D, if*

$$(Hx, x)\geqq\|x\|^2 \qquad \text{for every } x\in D,$$

then we can find a positive self-adjoint operator $\widetilde{H}\supset H$.

Proof. From assumption we conclude for every $x\in D$

$$\|x\|^2\leqq(Hx,x)\leqq\|Hx\|\|x\|,$$

and hence

$$\|Hx\|\geqq\|x\| \qquad \text{for every } x\in D.$$

Since H is closed, we see therefore that the range HD is a closed linear manifold. The orthogonal complement $(HD)^\perp$ is linearly independent from D, that is,

$$u + x = 0,\quad u\in(HD)^\perp,\quad x\in D$$

implies $u = x = 0$. Because for $x, y\in D$, $u\in(HD)^\perp$ we have

$$(Hy, u+x) = (Hy, u) + (Hy, x) = (y, Hx).$$

Thus if $u+x = 0$, then we have $(y, Hx) = 0$ for every $y \in D$, and hence $Hx = 0$. This relation yields $x = 0$, because

$$\|Hx\| \geqq \|x\| \qquad \text{for every } x \in D.$$

Therefore $(HD)^{\perp}$ is linearly independent from D.

Let $\widetilde{D}$ be the linear manifold generated by D and $(HD)^{\perp}$. Then every element $\widetilde{x} \in \widetilde{D}$ may be represented uniquely in the form

$$\widetilde{x} = u+x, \quad u \in (HD)^{\perp}, \quad x \in D,$$

because $(HD)^{\perp}$ is linearly independent from D, as proved just above. Thus putting

$$\widetilde{H}\widetilde{x} = Hx \quad \text{for } \widetilde{x} = u+x, \ u \in (HD)^{\perp}, \ x \in D,$$

we obtain an operator $\widetilde{H}$ with domain $\widetilde{D}$. It is obvious that $\widetilde{H} \supset H$ and $\widetilde{H}$ is closed.

For some $y, z \in R$, if

$$(\widetilde{H}(u+x), y) = (u+x, z) \qquad \text{for } u \in (HD)^{\perp}, \ x \in D,$$

then we have

$$(Hx, y) = (u, z) + (x, z) \qquad \text{for every } u \in (HD)^{\perp}, \ x \in D.$$

Thus putting $x = 0$, we obtain

$$(u, z) = 0 \qquad \text{for every } u \in (HD)^{\perp},$$

and hence $z \in HD$, because HD is a closed linear manifold, as proved just above. Therefore there exists an element $x_1 \in D$ for which

$$z = Hx_1.$$

For such $x_1 \in D$, we have then

$$(Hx, y) = (x, Hx_1) = (Hx, x_1),$$

and hence

$$(Hx, y - x_1) = 0 \qquad \text{for every } x \in D.$$

Thus putting $u_1 = y - x_1$, we have $u_1 \in (HD)^{\perp}$, and

$$z = Hx_1 = \widetilde{H}(u_1 + x_1) = \widetilde{H}y.$$

Therefore $\widetilde{H}$ is self-adjoint. Furthermore we have

$$(\widetilde{H}\widetilde{x}, \widetilde{x}) = (Hx, u+x) = (Hx, x) \geqq \|x\|^2 \geqq 0$$

for $\widetilde{x} = u+x$, $u \in (HD)^{\perp}$, $x \in D$.

Theorem 40.5. *For a Hermitean operator H with domain D, if*

$$(Hx, x) \geqq \alpha \|x\|^2 \qquad \text{for every } x \in D,$$

then for any real number $\beta < \alpha$ we can find a self-adjoint operator $\widetilde{H}$

with domain $\widetilde{D}$ *such that* $\widetilde{H} \supset H$ *and*

$$(\widetilde{H}x, x) \geqq \beta \|x\|^2 \qquad \text{for every } x \in \widetilde{D}.$$

Proof. Putting

$$H_1 = \frac{1}{\alpha - \beta}(H - \beta),$$

we obtain obviously a Hermitean operator H_1 with the same domain D and we have

$$(H_1 x, x) \geqq \|x\|^2 \qquad \text{for every } x \in D.$$

Thus we can find by Theorem 40.4 a positive self-adjoint operator $\widetilde{H}_1 \supset H_1$. For such $\widetilde{H}_1$, putting

$$\widetilde{H} = (\alpha - \beta)\widetilde{H}_1 + \beta,$$

we also obtain a self-adjoint operator $\widetilde{H}$, and we see easily that the domain of $\widetilde{H}$ coincides with that of $\widetilde{H}_1$ and $\widetilde{H} \supset H$. Denoting by $\widetilde{D}$ the domain of $\widetilde{H}$, we have furthermore for every $x \in \widetilde{D}$

$$\begin{aligned}(\widetilde{H}x, x) &= (\alpha - \beta)(\widetilde{H}_1 x, x) + \beta \|x\|^2 \\ &\geqq \beta \|x\|^2.\end{aligned}$$

CHAPTER V

UNITARY INVARIANTS

§41 Isomorphisms of Rings

For two rings of projection operators $\mathfrak{R}$ and $\mathfrak{T}$ in a Hilbert space $\mathfrak{H}$, if there is a one-to-one correspondence $P^{\mathfrak{T}} \in \mathfrak{T}$ $(P \in \mathfrak{R})$ such that

$$P^{\mathfrak{T}} \geqq Q^{\mathfrak{T}} \quad \text{if and only if} \quad P \geqq Q,$$
$$(P - Q)^{\mathfrak{T}} = P^{\mathfrak{T}} - Q^{\mathfrak{T}} \qquad \text{for } P \geqq Q,$$

then $\mathfrak{R}$ is said to be <u>isomorphic</u> to $\mathfrak{T}$. Such a one-to-one correspondence $P^{\mathfrak{T}} \in \mathfrak{T}$ $(P \in \mathfrak{R})$ is called an <u>isomorphism.</u>

Concerning isomorphisms $P^{\mathfrak{T}} \in \mathfrak{T}$ $(P \in \mathfrak{R})$ we have obviously by definition

$$(P \cup Q)^{\mathfrak{T}} = P^{\mathfrak{T}} \cup Q^{\mathfrak{T}}$$
$$(P \cap Q)^{\mathfrak{T}} = P^{\mathfrak{T}} \cap Q^{\mathfrak{T}}$$
$$0^{\mathfrak{T}} = 0.$$

Furthermore we see easily by definition that if a closed ring $\mathfrak{R}$ is isomorphic to a ring $\mathfrak{T}$ by an isomorphism $P^{\mathfrak{T}} \in \mathfrak{T}$ $(P \in \mathfrak{R})$, then $\mathfrak{T}$ also is closed and

$$\Big(\bigcup_{\lambda \in \Lambda} P_\lambda\Big)^{\mathfrak{T}} = \bigcup_{\lambda \in \Lambda} P_\lambda^{\mathfrak{T}}$$
$$\Big(\bigcap_{\lambda \in \Lambda} P_\lambda\Big)^{\mathfrak{T}} = \bigcap_{\lambda \in \Lambda} P_\lambda^{\mathfrak{T}}$$

for every system $P_\lambda \in \mathfrak{R}$ $(\lambda \in \Lambda)$, and hence for the cover $\mathfrak{C}_{\mathfrak{R}}$ of $\mathfrak{R}$ and $\mathfrak{C}_{\mathfrak{T}}$ of $\mathfrak{T}$ we have

$$\mathfrak{C}_{\mathfrak{R}}^{\mathfrak{T}} = \mathfrak{C}_{\mathfrak{T}}.$$

Let $\mathfrak{R}$ and $\mathfrak{T}$ be two closed rings of projection operators in the sequel. If $\mathfrak{R}$ is isomorphic to $\mathfrak{T}$ by an isomorphism $P^{\mathfrak{T}} \in \mathfrak{T}$ $(P \in \mathfrak{R})$, then for each maximal ideal $\mathfrak{P}$ in $\mathfrak{R}$, putting

$$\mathfrak{P}^{\mathfrak{T}} = \{ P^{\mathfrak{T}} : \ P \in \mathfrak{P} \},$$

we obtain a maximal ideal $\mathfrak{P}^{\mathfrak{T}}$ in $\mathfrak{T}$. Conversely for each maximal ideal $\mathfrak{q}$ in $\mathfrak{T}$, putting

$$\mathfrak{P} = \{ P : \ P^{\mathfrak{T}} \in \mathfrak{q} \},$$

we obtain a maximal ideal $\mathfrak{P}$ in $\mathfrak{R}$. Therefore we have

Theorem 41.1. *If $\mathcal{R}$ is isomorphic to $\mathcal{T}$ by an isomorphism*

$$P^{\mathcal{T}} \in \mathcal{T} \qquad (P \in \mathcal{R}),$$

then the proper space $\mathcal{C}_{\mathcal{R}}$ of $\mathcal{R}$ is homeomorphic to the proper space $\mathcal{C}_{\mathcal{T}}$ of $\mathcal{T}$ by a correspondence

$$\mathfrak{F}^{\mathcal{T}} \in \mathcal{C}_{\mathcal{T}} \qquad (\mathfrak{F} \in \mathcal{C}_{\mathcal{R}})$$

such that $\mathfrak{F}^{\mathcal{T}} \in U_P \mathcal{T}$ if and only if $\mathfrak{F} \in U_P$.

Conversely we have

Theorem 41.2. *If the proper space $\mathcal{C}_{\mathcal{R}}$ of $\mathcal{R}$ is homeomorphic to the proper space $\mathcal{C}_{\mathcal{T}}$ of $\mathcal{T}$ by a correspondence*

$$\mathfrak{F}^{\mathcal{T}} \in \mathcal{C}_{\mathcal{T}} \qquad (\mathfrak{F} \in \mathcal{C}_{\mathcal{R}}),$$

then $\mathcal{R}$ is isomorphic to $\mathcal{T}$ by an isomorphism

$$P^{\mathcal{T}} \in \mathcal{T} \qquad (P \in \mathcal{R})$$

such that $\mathfrak{F}^{\mathcal{T}} \in U_P \mathcal{T}$ if and only if $\mathfrak{F} \in U_P$.

Proof. By virtue of Theorem 15.6, putting

$$U_P \mathcal{T} = \{ \mathfrak{F}^{\mathcal{T}} : \ \mathfrak{F} \in U_P \} \qquad (P \in \mathcal{R}),$$

we obtain a one-to-one correspondence $P^{\mathcal{T}} \in \mathcal{T}$ $(P \in \mathcal{R})$. Furthermore it is obvious that $C_{\mathcal{R}}^{\mathcal{T}} = C_{\mathcal{T}}$, and

$$(C_{\mathcal{R}} - P)^{\mathcal{T}} = C_{\mathcal{T}} - P^{\mathcal{T}},$$

because we have by the formula §15(6)

$$\begin{aligned} U_{(C_{\mathcal{R}} - P)} \mathcal{T} &= \{ \mathfrak{F}^{\mathcal{T}} : \ \mathfrak{F} \in U_{C_{\mathcal{R}} - P} \} \\ &= \{ \mathfrak{F}^{\mathcal{T}} : \ \mathfrak{F} \in U_P \}' = U_{C_{\mathcal{T}} - P^{\mathcal{T}}}. \end{aligned}$$

If $P = \bigcup_{\lambda \in \Lambda} P_\lambda$, $P_\lambda \in \mathcal{R}$ $(\lambda \in \Lambda)$, then we have by Theorem 15.7

$$U_P = (\sum_{\lambda \in \Lambda} U_{P_\lambda})^-.$$

This relation yields

$$U_P \mathcal{T} = \{ \mathfrak{F}^{\mathcal{T}} : \ \mathfrak{F} \in \sum_{\lambda \in \Lambda} U_{P_\lambda} \}^- = (\sum_{\lambda \in \Lambda} U_{P_\lambda} \mathcal{T})^-,$$

and hence $P^{\mathcal{T}} = \bigcup_{\lambda \in \Lambda} P_\lambda^{\mathcal{T}}$ by Theorem 15.7. Thus $P^{\mathcal{T}} \in \mathcal{T}$ $(P \in \mathcal{R})$ is an isomorphism.

For two closed linear manifolds M and N, a mapping

$$x^N \in N \qquad (x \in M)$$

of M onto N is said to be an isometric correspondence, if

$$(\alpha x + \beta y)^N = \alpha x^N + \beta y^N,$$

$$(x, y) = (x^N, y^N)$$

for every x, $y \in M$ and complex numbers α, β.

For an isometric correspondence $x^N \in N$ $(x \in M)$ we have obviously by definition

$$\| x^N \| = \| x \| \qquad \text{for every } x \in M,$$

$$x^N = 0 \quad \text{if and only if} \quad x = 0,$$

$$\lim_{\nu \to \infty} x_\nu^N = x^N \quad \text{if and only if} \quad \lim_{\nu \to \infty} x_\nu = x.$$

Furthermore, putting

$$U x = (P_M x)^N \qquad (x \in R)$$

for the projection operator P_M of M, we obtain by definition an isometric operator U with proper domain M and range N.

Let U be an isometric operator with proper domain D. By virtue of Theorem 38.1, for the adjoint operator U^* of U and the projection operator P_D of D we have

$$U^* U = P_D,$$

and hence

$$(U x, U y) = (x, y) \qquad \text{for every } x, y \in D.$$

Thus $U x$ $(x \in D)$ is an isometric correspondence between the proper domain D and the range UD. For a manifold $K \subset D$, we see easily that UK is a closed linear manifold, if and only if K is a closed linear manifold. Furthermore for the projection operator P_K of a closed linear manifold $K \subset D$ and the projection operator P_{UK} of UK, we have

$$P_{UK} U x = U P_K x \qquad \text{for every } x \in R.$$

Since $U U^* = P_{UD}$ by Theorem 38.1, we obtain hence

$$(1) \qquad P_{UK} = U P_K U^* \qquad \text{for } K \subset D.$$

From this relation we conclude immediately for every system of closed linear manifolds $K_\lambda \subset D$ $(\lambda \in \Lambda)$

$$(2) \qquad \bigcup_{\lambda\in\Lambda} U P_{K_\lambda} U^* = U\left(\bigcup_{\lambda\in\Lambda} P_{K_\lambda}\right) U^*,$$

$$(3) \qquad \bigcap_{\lambda\in\Lambda} U P_{K_\lambda} U^* = U\left(\bigcap_{\lambda\in\Lambda} P_{K_\lambda}\right) U^*.$$

For two closed linear manifolds $K, L \subset D$ we have obviously

$$(U P_K U^*)(U P_L U^*) = U P_K P_L U^*,$$

$$U^*(U P_K P_L U^*) U = P_K P_L,$$

because $U^* U = P_D$. Thus we see that

$$(U P_K U^*)(U P_L U^*) = (U P_L U^*)(U P_K U^*)$$

if and only if $P_K P_L = P_L P_K$.

Therefore, if $C_{\mathcal{R}} \leqq P_D$ for the cover $C_{\mathcal{R}}$ of a ring of projection operators $\mathcal{R}$, then putting

$$\mathcal{T} = \{ U P U^* : \quad P \in \mathcal{R} \},$$

we obtain a ring of projection operators $\mathcal{T}$, and $\mathcal{R}$ is isomorphic to $\mathcal{T}$ by the correspondence $U P U^* \in \mathcal{T}$ $(P \in \mathcal{R})$. Furthermore for the commutor $\overline{\mathcal{R}}$ of $\mathcal{R}$ and $\overline{\mathcal{T}}$ of $\mathcal{T}$, we see easily that

$$U P U^* \in C_{\mathcal{T}} \overline{\mathcal{T}} \quad (P \in C_{\mathcal{R}} \overline{\mathcal{R}})$$

is an isomorphism between $C_{\mathcal{R}} \overline{\mathcal{R}}$ and $C_{\mathcal{T}} \overline{\mathcal{T}}$.

Theorem 41.3. *For two closed rings of projection operators $\mathcal{R}$, $\mathcal{T}$ and the commutors $\overline{\mathcal{R}}$, $\overline{\mathcal{T}}$ respectively of $\mathcal{R}$, $\mathcal{T}$, if there is a one-to-one correspondence*

$$P^{\mathcal{T}} \in C_{\mathcal{T}} \overline{\mathcal{T}} \quad (P \in C_{\mathcal{R}} \overline{\mathcal{R}})$$

between $C_{\mathcal{R}} \overline{\mathcal{R}}$ and $C_{\mathcal{T}} \overline{\mathcal{T}}$ such that

$$P^{\mathcal{T}} \geqq Q^{\mathcal{T}} \text{ if and only if } P \geqq Q,$$

$$(P - Q)^{\mathcal{T}} = P^{\mathcal{T}} - Q^{\mathcal{T}} \text{ for } P \geqq Q,$$

then there exists an isometric operator U with proper domain $C_{\mathcal{R}} R$ such that $\mathcal{R}$ is isomorphic to $\mathcal{T}$ by the correspondence

$$U P U^* \in \mathcal{T} \qquad (P \in \mathcal{R})$$

and we have

$$P^{\mathcal{T}} = U P U^* \qquad \text{for every } P \in \mathcal{R}.$$

Proof. We see easily by assumption that

$$\left(\bigcup_{\lambda\in\Lambda} P_\lambda\right)^{\mathcal{T}} = \bigcup_{\lambda\in\Lambda} P_\lambda^{\mathcal{T}},$$

$$(\bigwedge_{\lambda\in\Lambda} P_\lambda)^{\mathfrak{T}} = \bigwedge_{\lambda\in\Lambda} P_\lambda^{\mathfrak{T}}$$

for every system $P_\lambda \in C_{\mathfrak{R}}\overline{\mathfrak{R}}$ $(\lambda \in \Lambda)$. For $P, Q \in C_{\mathfrak{R}}\overline{\mathfrak{R}}$, if

$$PQ = QP,$$

then we have by Theorem 12.6

$$(P \vee Q) - P = Q - (P \wedge Q),$$

and hence by assumption

$$(P^{\mathfrak{T}} \vee Q^{\mathfrak{T}}) - P^{\mathfrak{T}} = Q^{\mathfrak{T}} - (P^{\mathfrak{T}} \wedge Q^{\mathfrak{T}}),$$

that is,

$$P^{\mathfrak{T}} \vee Q^{\mathfrak{T}} = P^{\mathfrak{T}} + (Q^{\mathfrak{T}} - (P^{\mathfrak{T}} \wedge Q^{\mathfrak{T}})).$$

Thus we have by Theorem 12.4

$$P^{\mathfrak{T}}(Q^{\mathfrak{T}} - (P^{\mathfrak{T}} \wedge Q^{\mathfrak{T}})) = (Q^{\mathfrak{T}} - (P^{\mathfrak{T}} \wedge Q^{\mathfrak{T}}))P^{\mathfrak{T}} = 0.$$

Since we have by Theorem 12.1

$$P^{\mathfrak{T}}(P^{\mathfrak{T}} \wedge Q^{\mathfrak{T}}) = (P^{\mathfrak{T}} \wedge Q^{\mathfrak{T}})P^{\mathfrak{T}} = P^{\mathfrak{T}} \wedge Q^{\mathfrak{T}},$$

we obtain hence

$$P^{\mathfrak{T}}Q^{\mathfrak{T}} = Q^{\mathfrak{T}}P^{\mathfrak{T}}.$$

We also can prove likewise that $P^{\mathfrak{T}}Q^{\mathfrak{T}} = Q^{\mathfrak{T}}P^{\mathfrak{T}}$ implies $PQ = QP$.

Therefore we conclude by Theorem 20.1 that we have $P^{\mathfrak{T}} \in \mathfrak{T}$ if and only if $P \in \mathfrak{R}$. Thus $\mathfrak{R}$ is isomorphic to $\mathfrak{T}$ by this correspondence $P^{\mathfrak{T}} \in \mathfrak{T}$ $(P \in \mathfrak{R})$, and hence we see by Theorem 41.1 that the proper space $\mathfrak{L}_{\mathfrak{R}}$ of $\mathfrak{R}$ is homeomorphic to the proper space $\mathfrak{L}_{\mathfrak{T}}$ of $\mathfrak{T}$ by a correspondence

$$\mathfrak{x}^{\mathfrak{T}} \in \mathfrak{L}_{\mathfrak{T}} \qquad (\mathfrak{x} \in \mathfrak{L}_{\mathfrak{R}})$$

such that $\mathfrak{x}^{\mathfrak{T}} \in \mathfrak{U}_P$ if and only if $\mathfrak{x} \in \mathfrak{U}_P$.

By virtue of Theorem 14.1, for any $0 \neq P \in C_{\mathfrak{R}}\overline{\mathfrak{R}}$ we can find an element $a \in R$ such that $P \geqq [a]_{\mathfrak{R}} \neq 0$. For such $a \in R$, since $[a]_{\mathfrak{R}}^{\mathfrak{T}} \in C_{\mathfrak{T}}\overline{\mathfrak{T}}$, we can find $b \in R$ such that $[a]_{\mathfrak{R}}^{\mathfrak{T}} \geqq [b]_{\mathfrak{T}} \neq 0$. Then there exists $Q \in C_{\mathfrak{R}}\overline{\mathfrak{R}}$ for which $[b]_{\mathfrak{T}} = Q^{\mathfrak{T}}$, and hence we have $[a]_{\mathfrak{R}} \geqq Q \neq 0$. Thus we have by the formula §14(6)

$$[Qa]_{\mathfrak{R}} = Q[a]_{\mathfrak{R}} = Q \neq 0.$$

Therefore for any $0 \neq P \in C_{\mathfrak{R}}\overline{\mathfrak{R}}$ we can find $x, y \in R$ such that

$$P \geqq [x]_{\mathfrak{R}} \neq 0, \qquad [x]_{\mathfrak{R}}^{\mathfrak{T}} = [y]_{\mathfrak{T}}.$$

By virtue of Maximal Theorem, there exists a maximal system of elements $x_\lambda, y_\lambda \in R$ $(\lambda \in \Lambda)$ such that

$$[y_\lambda]_{\mathfrak{T}} = [x_\lambda]_{\mathfrak{R}}^{\mathfrak{T}} \neq 0,$$

$$[x_\lambda]_{\mathfrak{R}}[x_\rho]_{\mathfrak{R}} = 0 \qquad \text{for } \lambda \neq \rho.$$

Then we have $C_{\mathfrak{R}} = \bigcup_{\lambda\in\Lambda} [x_\lambda]_{\mathfrak{R}}$. Because, putting

$$P = C_{\mathfrak{R}} - \bigcup_{\lambda\in\Lambda} [x_\lambda]_{\mathfrak{R}},$$

if $P \neq 0$, then, since $P \in C_{\mathfrak{R}}\overline{\mathfrak{R}}$, we can find $x, y \in R$ for which

$$P \geqq [x]_{\mathfrak{R}} \neq 0, \quad [x]_{\mathfrak{R}}^{\mathfrak{r}} = [y]_{\mathfrak{r}},$$

and hence

$$[x_\lambda]_{\mathfrak{R}}[x]_{\mathfrak{R}} = 0 \qquad \text{for every } \lambda \in \Lambda,$$

contradicting the maximality of x_λ, $y_\lambda \in R$ $(\lambda \in \Lambda)$.

Therefore we have

$$C_{\mathfrak{R}} = \bigcup_{\lambda\in\Lambda} [x_\lambda]_{\mathfrak{R}}, \qquad C_{\mathfrak{r}} = \bigcup_{\lambda\in\Lambda} [y_\lambda]_{\mathfrak{r}}.$$

Since $[y_\lambda]_{\mathfrak{r}} = [x_\lambda]_{\mathfrak{R}}^{\mathfrak{r}}$, we have by definition

$$(C^{\mathfrak{R}}_{[x_\lambda]_{\mathfrak{R}}})^{\mathfrak{r}} = \Big(\bigwedge_{[x_\lambda]_{\mathfrak{R}} \leqq P \in \mathfrak{R}} P\Big)^{\mathfrak{r}} = \bigwedge_{[y_\lambda]_{\mathfrak{r}} \leqq P^{\mathfrak{r}} \in \mathfrak{r}} P^{\mathfrak{r}} = C^{\mathfrak{r}}_{[y_\lambda]_{\mathfrak{r}}}.$$

Since both $\| P x_\lambda \|^2$, $\| P^{\mathfrak{r}} y_\lambda \|^2$ $(P \in \mathfrak{R})$ are positive totally additive functions on $\mathfrak{R}$ and proper in $C^{\mathfrak{R}}_{[x_\lambda]_{\mathfrak{R}}}$, by virtue of Theorem 18.2, putting

$$\varphi_\lambda(\mathfrak{P}) = \lim_{P \to \mathfrak{P}} \frac{\| P^{\mathfrak{r}} y_\lambda \|^2}{\| P x_\lambda \|^2} \qquad \text{for } \mathfrak{P} \in \mathfrak{U}_{C^{\mathfrak{R}}_{[x_\lambda]_{\mathfrak{R}}}},$$

we obtain a positive continuous function φ_λ on $\mathfrak{U}_{C^{\mathfrak{R}}_{[x_\lambda]_{\mathfrak{R}}}}$ and

$$\| P^{\mathfrak{r}} y_\lambda \|^2 = \int_P \varphi_\lambda(\mathfrak{P}) \| d\mathfrak{P}\, x_\lambda \|^2 \qquad \text{for } \mathfrak{R} \ni P \leqq C^{\mathfrak{R}}_{[x_\lambda]_{\mathfrak{R}}}.$$

Thus we can put by Theorem 17.3

$$z_\lambda = \int_{C^{\mathfrak{R}}_{[x_\lambda]_{\mathfrak{R}}}} \sqrt{\varphi_\lambda(\mathfrak{P})}\, d\mathfrak{P}\, x_\lambda$$

and we have

$$\| P^{\mathfrak{r}} y_\lambda \| = \| P z_\lambda \| \qquad \text{for } \mathfrak{R} \ni P \leqq C^{\mathfrak{R}}_{[x_\lambda]_{\mathfrak{R}}}.$$

Furthermore we have $[z_\lambda]_{\mathfrak{R}} = [x_\lambda]_{\mathfrak{R}}$. Because it is obvious that $[z_\lambda]_{\mathfrak{R}} \leqq [x_\lambda]_{\mathfrak{R}}$, and hence we can find by Theorem 19.4 $P_0 \in \mathfrak{R}$ such that $P_0 \leqq C^{\mathfrak{R}}_{[x_\lambda]_{\mathfrak{R}}}$ and $[z_\lambda]_{\mathfrak{R}} = P_0[x_\lambda]_{\mathfrak{R}}$. Then we have

$$\| (C^{\mathfrak{r}}_{[y_\lambda]_{\mathfrak{r}}} - P_0^{\mathfrak{r}}) y_\lambda \| = \| (C^{\mathfrak{R}}_{[x_\lambda]_{\mathfrak{R}}} - P_0)^{\mathfrak{r}} y_\lambda \|$$
$$= \| (C^{\mathfrak{R}}_{[x_\lambda]_{\mathfrak{R}}} - P_0) z_\lambda \| = 0.$$

This relation yields

$$P_0^{\mathfrak{r}} = C^{\mathfrak{r}}_{[y_\lambda]_{\mathfrak{r}}} = (C^{\mathfrak{R}}_{[x_\lambda]_{\mathfrak{R}}})^{\mathfrak{r}},$$

and hence $P_0 = C^{\mathfrak{R}}_{[x_\lambda]_{\mathfrak{R}}}$.

For every $x \in [z_\lambda]_{\mathfrak{R}} R$, putting

$$U x = \int_{C^{\mathfrak{T}}_{[y_\lambda]_{\mathfrak{T}}}} \left(\frac{x}{z_\lambda}, \xi\right) d_\xi E^{\mathfrak{T}}_\xi y_\lambda,$$

we see by Theorems 17.3 and 18.4 that $[z_\lambda]_{\mathfrak{R}} R$ is isometric to the closed linear manifold $[y_\lambda]_{\mathfrak{T}} R$ by the correspondence

$$U x \in [y_\lambda]_{\mathfrak{T}} R \quad (x \in [z_\lambda]_{\mathfrak{R}} R).$$

Since we have by the formulas (6) and (8) in §19

$$U P x = \int_{P^{\mathfrak{T}}} \left(\frac{x}{z_\lambda}, \xi\right) d_\xi E^{\mathfrak{T}}_\xi y_\lambda \qquad \text{for } \mathfrak{R} \ni P \leqq C^{\mathfrak{R}}_{[z_\lambda]_{\mathfrak{R}}},$$

we conclude by Theorem 17.4

$$U P [z_\lambda]_{\mathfrak{R}} R = P^{\mathfrak{T}} [y_\lambda]_{\mathfrak{T}} R \qquad \text{for every } P \in \mathfrak{R},$$

because $P[z_\lambda]_{\mathfrak{R}} = P\, C^{\mathfrak{R}}_{[z_\lambda]_{\mathfrak{R}}} [z_\lambda]_{\mathfrak{R}}$.

For any $x \in R$ we can find a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ such that

$$C_{\mathfrak{R}} x = \sum_{\nu=1}^{\infty} [z_{\lambda_\nu}]_{\mathfrak{R}} x,$$

and for such λ_ν, putting

$$U x = \sum_{\nu=1}^{\infty} U [z_{\lambda_\nu}]_{\mathfrak{R}} x,$$

we obtain hence an isometric operator U with proper domain $C_{\mathfrak{R}} R$, and we also have for each $\lambda \in \Lambda$

$$U P [z_\lambda]_{\mathfrak{R}} R = P^{\mathfrak{T}} [y_\lambda]_{\mathfrak{T}} R \qquad \text{for every } P \in \mathfrak{R},$$

and hence by the formula (1)

$$U (P [z_\lambda]_{\mathfrak{R}}) U^* = P^{\mathfrak{T}} [y_\lambda]_{\mathfrak{T}} \qquad \text{for every } \lambda \in \Lambda, P \in \mathfrak{R}.$$

Since

$$\bigcup_{\lambda \in \Lambda} [z_\lambda]_{\mathfrak{R}} = C_{\mathfrak{R}},$$

$$\bigcup_{\lambda \in \Lambda} [y_\lambda]_{\mathfrak{T}} = \bigcup_{\lambda \in \Lambda} [z_\lambda]^{\mathfrak{T}}_{\mathfrak{R}} = C^{\mathfrak{T}}_{\mathfrak{R}} = C_{\mathfrak{T}},$$

we obtain therefore by the formula (2)

$$U P U^* = P^{\mathfrak{T}} \qquad \text{for every } P \in \mathfrak{R}.$$

If $\mathfrak{R}$ is simple, then we have by definition $\mathfrak{R} = C_{\mathfrak{R}} \overline{\mathfrak{R}}$, and hence we obtain immediately by Theorem 4...3

Theorem 41.4 *Let a simple ring of projection operators $\mathfrak{R}$ be isomorphic to a ring $\mathfrak{T}$ by a correspondence $P \to P^{\mathfrak{T}} \in \mathfrak{T}$ $(P \in \mathfrak{R})$. In order that we can find an isometric operator U with proper domain $C_{\mathfrak{R}} R$ such that*

$$P^{\gamma} = UPU^{*} \qquad \text{for every } P \in \mathfrak{R},$$

it is necessary and sufficient that γ also is simple.

§42 Rings of Uniform Multiplicity.

Let $\mathfrak{R}$ be a closed ring of projection operators and $\overline{\mathfrak{R}}$ the commutor of $\mathfrak{R}$. For every $Q \in \overline{\mathfrak{R}}$, $Q\mathfrak{R}$ also is obviously a closed ring. For a projection operator Q we have by the definition in §14

$$C_{Q} = C_{\mathfrak{R}}$$

if and only if $QP = 0$, $P \in \mathfrak{R}$, implies $P = 0$.

Theorem 42.1. *For a projection operator $Q \in \overline{\mathfrak{R}}$, $\mathfrak{R}$ is isomorphic to $Q\mathfrak{R}$ by the correspondence $QP \in Q\mathfrak{R}$ $(P \in \mathfrak{R})$, if and only if $C_{Q} = C_{\mathfrak{R}}$.*

Proof. For $P_1, P_2 \in \mathfrak{R}$ we have obviously that

$$P_1 \leqq P_2 \quad \text{implies} \quad QP_1 \leqq QP_2.$$

If $C_{Q} = C_{\mathfrak{R}}$, then we have conversely that

$$QP_1 \leqq QP_2 \quad \text{implies} \quad P_1 \leqq P_2.$$

In fact, if $QP_1 \leqq QP_2$, then we have $Q(P_1 - P_1P_2) = 0$, and hence $P_1 - P_1P_2 = 0$, because $C_{Q} = C_{\mathfrak{R}}$ and $P_1 - P_1P_2 \in \mathfrak{R}$.

If $C_{Q} \neq C_{\mathfrak{R}}$, then putting $P_0 = C_{\mathfrak{R}} - C_{Q}$, we have

$$0 \neq P_0 \in \mathfrak{R}$$

but

$$QP_0 = Q \bigvee_{QP=0,\, P \in \mathfrak{R}} P = 0.$$

A projection operator $S \in C_{\mathfrak{R}}\overline{\mathfrak{R}}$ is said to be a *simplor* of $\mathfrak{R}$, if $C_{S} = C_{\mathfrak{R}}$ and $S\mathfrak{R}$ is a simple ring.

Theorem 42.2. *For a projection operator $Q \in \overline{\mathfrak{R}}$, if*

$$C_{Q} = C_{\mathfrak{R}},$$

then there exists a simplor $S \leqq Q$ of $\mathfrak{R}$.

Proof. By virtue of Maximal Theorem, we can find a maximal system of elements $x_{\lambda} \in R$ $(\lambda \in \Lambda)$ subject ot the condition:

$$Qx_{\lambda} \neq 0 \qquad \text{for every } \lambda \in \Lambda,$$

$$C_{[Qx_{\lambda}]}\, C_{[Qx_{\rho}]} = 0 \qquad \text{for } \lambda \neq \rho.$$

For such $x_\lambda \in R$ $(\lambda \in \Lambda)$, we have

$$\bigcup_{\lambda\in\Lambda} C_{[Qx_\lambda]} = C_{\mathcal{R}}.$$

Because putting

$$P_0 = C_{\mathcal{R}} - \bigcup_{\lambda\in\Lambda} C_{[Qx_\lambda]},$$

if $P_0 \neq 0$, then we have $P_0 Q \neq 0$ by assumption, and for an element $x \in R$ such that $P_0 Q x \neq 0$, we have by the formula §14(16') for every $\lambda \in \Lambda$

$$C_{[Qx_\lambda]} C_{[P_0 Qx]} = C_{[Qx_\lambda]} P_0 C_{[Qx]} = 0,$$

contradicting the maximality of the system $x_\lambda \in R$ $(\lambda \in \Lambda)$.

For such $x_\lambda \in R$ $(\lambda \in \Lambda)$, putting

$$S = \bigcup_{\lambda\in\Lambda} [Qx_\lambda],$$

we have obviously $C_{\mathcal{R}}\overline{\mathcal{R}} \ni S \leqq Q$, and by the formula §14(15)

$$C_S = \bigcup_{\lambda\in\Lambda} C_{[Qx_\lambda]} = C_{\mathcal{R}}.$$

For a projection operator Q_0, if

$$(SP)Q_0 = Q_0(SP) = 0 \qquad \text{for } 0 \neq P \in \mathcal{R},$$

then we have naturally for $C_{[Qx_\lambda]} P \neq 0$, $P \in \mathcal{R}$,

$$(S C_{[Qx_\lambda]} P)Q_0 = Q_0(S C_{[Qx_\lambda]} P) \neq 0 \qquad \text{for every } \lambda \in \Lambda,$$

because $C_{[Qx_\lambda]} \in \mathcal{R}$. On the other hand, since

$$C_{[Qx_\lambda]} C_{[Qx_\rho]} = 0 \qquad \text{for } \lambda \neq \rho,$$

we have

$$S\, C_{[Qx_\lambda]} = [Qx_\lambda],$$

and $[Qx_\lambda]\mathcal{R}$ is simple by Theorem 20.5. Thus we obtain by Theorem 20.4 $Q_0 \geqq [Qx_\lambda]$ for every $\lambda \in \Lambda$, and consequently

$$Q_0 \geqq \bigcup_{\lambda\in\Lambda} [Qx_\lambda] = S.$$

Therefore $S\mathcal{R}$ is simple by Theorem 20.4.

A system of simplors S_λ $(\lambda \in \Lambda)$ of $\mathcal{R}$ is said to be <u>orthogonal</u>, if $S_\lambda S_\rho = 0$ for $\lambda \neq \rho$. An orthogonal system of simplors S_λ $(\lambda \in \Lambda)$ of $\mathcal{R}$ is said to be <u>complete</u>, if

$$C_{\mathcal{R}} = \bigcup_{\lambda\in\Lambda} S_\lambda$$

for the cover $C_{\mathcal{R}}$ of $\mathcal{R}$. A closed ring of projection operators $\mathcal{R}$ is said to be of <u>uniform multiplicity</u>, if there exists a complete orthogonal system of simplors of $\mathcal{R}$.

<u>Theorem</u> 42.3. <u>Let a closed ring of projection operators</u> $\mathcal{R}$ <u>be</u>

of uniform multiplicity and S_λ $(\lambda \in \Lambda)$ a complete orthogonal system of simplors of $\mathfrak{R}$. For any orthogonal system of simplors Q_γ $(\gamma \in \Gamma)$ of $\mathfrak{R}$, the density of Γ is not greater than that of Λ.

Proof. For an element $a \in R$ subject to $C_{\mathfrak{R}} a \neq 0$, we have obviously $C_{[a]} \neq 0$ by the formula §14(16'). Since $\mathfrak{R}$ is isomorphic to $S_\lambda \mathfrak{R}$ by the correspondence

$$S_\lambda P \in S_\lambda \mathfrak{R} \quad (P \in \mathfrak{R}),$$

we see by Theorem 14.7 that there exists $x_\lambda \in R$ $(\lambda \in \Lambda)$ such that

$$C^{S_\lambda \mathfrak{R}}_{[x_\lambda]_{S_\lambda \mathfrak{R}}} = S_\lambda C_{[a]}.$$

On the other hand we have by the formula §20(1)

$$[S_\lambda x_\lambda] = [x_\lambda]_{S_\lambda \mathfrak{R}} = C^{S_\lambda \mathfrak{R}}_{[x_\lambda]_{S_\lambda \mathfrak{R}}},$$

because $S_\lambda \mathfrak{R}$ is simple. Thus for such $x_\lambda \in R$ $(\lambda \in \Lambda)$ we have

$$[S_\lambda x_\lambda] = S_\lambda C_{[a]} \qquad \text{for every } \lambda \in \Lambda.$$

We also can find likewise $y_\gamma \in R$ $(\gamma \in \Gamma)$ such that

$$[Q_\gamma y_\gamma] = Q_\gamma C_{[a]} \qquad \text{for every } \gamma \in \Gamma,$$

and hence by the formula §14(16)

$$C_{[Q_\gamma y_\gamma]} = C_{[a]} C_{Q_\gamma} = C_{[a]} C_{\mathfrak{R}} = C_{[a]}.$$

We suppose firstly that the density of Λ is infinite. For any $\lambda \in \Lambda$, we have

$$\| S_\lambda x_\lambda \|^2 \geqq \sum_{\gamma \in \Gamma} \| [Q_\gamma y_\gamma] S_\lambda x_\lambda \|^2,$$

and hence we can find a sequence $\gamma_{\lambda,\nu}$ $(\nu = 1, 2, \ldots)$ such that

$$[Q_\gamma y_\gamma] S_\lambda x_\lambda = 0 \qquad \text{for } \gamma \neq \gamma_{\lambda,\nu} \quad (\nu = 1, 2, \ldots).$$

On the other hand, for each $\gamma \in \Gamma$ we can find λ and ν for which $\gamma = \gamma_{\lambda,\nu}$, because if $[Q_\gamma y_\gamma] S_\lambda x_\lambda = 0$ for every $\lambda \in \Lambda$, then we have by §14(8)

$$[Q_\gamma y_\gamma][S_\lambda x_\lambda] = 0 \qquad \text{for every } \lambda \in \Lambda,$$

and hence

$$\begin{aligned}[Q_\gamma y_\gamma] &= [Q_\gamma y_\gamma] \bigcup_{\lambda \in \Lambda} S_\lambda C_{[a]} \\ &= \bigcup_{\lambda \in \Lambda} [Q_\gamma y_\gamma][S_\lambda x_\lambda] = 0,\end{aligned}$$

contradicting $[Q_\gamma y_\gamma] \neq 0$. Therefore we have

$$\Gamma = \{ \gamma_{\lambda,\nu} : \lambda \in \Lambda, \ \nu = 1, 2, \ldots \}.$$

Thus the density of Γ is not greater than that of Λ.

Next we consider the case where Λ is a finite set. We shall prove firstly that for every finite number of different elements $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots, \varkappa)$, if

$$C_\mu \in \left(\sum_{\nu=1}^{\varkappa} [S_{\lambda_\nu} x_{\lambda_\nu}]\right) R \qquad (\mu = 1, 2, \ldots, \varkappa+1),$$

then for any $0 \neq P \in \mathfrak{R}$ we can find dilatators T_μ $(\mu = 1, 2, \ldots, \varkappa+1)$ by $\mathfrak{R}$ such that

$$\sum_{\mu=1}^{\varkappa+1} T_\mu C_\mu = 0$$

but we have not $T_\mu P = 0$ for all $\mu = 1, 2, \ldots, \varkappa+1$. In the case: $\varkappa = 1$, if

$$C_1 = [S_{\lambda_1} x_{\lambda_1}] C_1, \quad C_2 = [S_{\lambda_1} x_{\lambda_1}] C_2,$$

then we have obviously

$$PC_1 = [S_{\lambda_1} x_{\lambda_1}] PC_1, \quad PC_2 = [S_{\lambda_1} x_{\lambda_1}] PC_2.$$

If $PC_1 = PC_2 = 0$, then we have $PC_1 + PC_2 = 0$ and $P \neq 0$. Thus we suppose that $PC_2 \neq 0$. By virtue of Theorem 25.8, we can find two dilatators T_1 and T_2 such that

$$PC_1 = T_1(S_{\lambda_1} x_{\lambda_1}), \quad PC_2 = T_2(S_{\lambda_1} x_{\lambda_1}), \quad [T_2] = C_{[PC_2]}.$$

For such T_1, T_2 we have by the formula §25(40)

$$[T_2] PC_1 - (T_1 \circ T_2^{-1}) PC_2 = 0$$

but $[T_2] P = P C_{[PC_2]} = C_{[PC_2]} \neq 0$. Next we suppose that it is valid in the case: $\varkappa - 1$. We have obviously by assumption

$$PC_\mu = \sum_{\nu=1}^{\varkappa} [S_{\lambda_\nu} x_{\lambda_\nu}] PC_\mu \qquad (\mu = 1, 2, \ldots, \varkappa+1).$$

If $[S_{\lambda_\varkappa} x_{\lambda_\varkappa}] PC_\mu = 0$ for every $\mu = 1, 2, \ldots, \varkappa+1$, then we have

$$PC_\mu \in \left(\sum_{\nu=1}^{\varkappa-1} [S_{\lambda_\nu} x_{\lambda_\nu}]\right) R \qquad (\mu = 1, 2, \ldots, \varkappa+1).$$

Thus we suppose that $[S_{\lambda_\varkappa} x_{\lambda_\varkappa}] PC_{\varkappa+1} \neq 0$. By virtue of Theorem 25.8, we can find dilatators $T_{\varkappa,\mu}$ $(\mu = 1, 2, \ldots, \varkappa+1)$ such that

$$[S_{\lambda_\varkappa} x_{\lambda_\varkappa}] PC_\mu = T_{\varkappa,\mu}(S_{\lambda_\varkappa} x_{\lambda_\varkappa}) \qquad (\mu = 1, 2, \ldots, \varkappa+1),$$

$$[T_{\varkappa,\varkappa+1}] = C_{[[S_{\lambda_\varkappa} x_{\lambda_\varkappa}] PC_{\varkappa+1}]},$$

and hence by §25(40)

$$[T_{\varkappa,\varkappa+1}] PC_\mu - (T_{\varkappa,\mu} \circ T_{\varkappa,\varkappa+1}^{-1}) PC_{\varkappa+1}$$
$$= \sum_{\nu=1}^{\varkappa-1} [S_{\lambda_\nu} x_{\lambda_\nu}] [T_{\varkappa,\varkappa+1}] PC_\mu \in \left(\sum_{\nu=1}^{\varkappa-1} [S_{\lambda_\nu} x_{\lambda_\nu}]\right) R.$$

Thus we can find by assumption dilatators T_μ $(\mu = 1, 2, \ldots, \varkappa)$ by $\mathfrak{R}$ such that

$$\sum_{\mu=1}^{\varkappa} T_\mu \{[T_{\varkappa,\varkappa+1}] PC_\mu - (T_{\varkappa,\mu} \circ T_{\varkappa,\varkappa+1}^{-1}) PC_{\varkappa+1}\} = 0$$

but $T_\mu[T_{\varkappa,\varkappa+1}]P \neq 0$ for some μ, because we have by §14(16')

$$[T_{\varkappa,\varkappa+1}]P = C_{[[S_{\lambda_\varkappa x_{\lambda_\varkappa}}]P C_{\varkappa+1}]} \neq 0.$$

For such T_μ $(\mu = 1, 2, \ldots, \varkappa)$ we can find by Theorem 24.4 $Q \in \mathfrak{R}$ for which all $T_\mu Q$ $(\mu = 1, 2, \ldots, \varkappa)$ are bounded and

$$T_\mu Q [T_{\varkappa,\varkappa+1}] P \neq 0 \qquad \text{for some } \mu.$$

Then we also have

$$\sum_{\mu=1}^{\varkappa} T_\mu Q [T_{\varkappa,\varkappa+1}] P C_\mu - \sum_{\mu=1}^{\varkappa} (T_\mu \circ T_{\varkappa,\mu} \circ T_{\varkappa,\varkappa+1}^{-1}) Q P C_{\varkappa+1} = 0$$

but $T_\mu Q [T_{\varkappa,\varkappa+1}]P \neq 0$ for some μ. Therefore we conclude our assertion by induction.

If $\Lambda = \{\lambda_\nu : \nu = 1, 2, \ldots, \varkappa\}$, then we have

$$\sum_{\nu=1}^{\varkappa} [S_{\lambda_\nu} x_{\lambda_\nu}] = C_{[a]}.$$

If we have different $\gamma_\nu \in \Gamma$ $(\nu = 1, 2, \ldots, \varkappa+1)$, then we can find dilatators T_μ $(\mu = 1, 2, \ldots, \varkappa+1)$ by $\mathfrak{R}$ such that

$$\sum_{\mu=1}^{\varkappa+1} T_\mu (Q_{\gamma_\mu} y_{\gamma_\mu}) = 0,$$

but $T_\mu C_{[a]} \neq 0$ for some μ. On the other hand we have for every $\nu = 1, 2, \ldots, \varkappa+1$

$$[Q_{\gamma_\nu} y_{\gamma_\nu}] \sum_{\mu=1}^{\varkappa+1} T_\mu (Q_{\gamma_\mu} y_{\gamma_\mu}) = T_\nu (Q_{\gamma_\nu} y_{\gamma_\nu}).$$

Thus we obtain by Theorem 25.7 for every $\mu = 1, 2, \ldots, \varkappa+1$

$$T_\mu C_{[a]} = T_\mu C_{[Q_{\gamma_\mu} y_{\gamma_\mu}]} = 0,$$

contradicting $T_\mu C_{[a]} \neq 0$ for some μ. Therefore the density of Γ is not greater than that of Λ.

By virtue of this Theorem 42.3, we see that for a closed ring $\mathfrak{R}$ of uniform multiplicity, every complete orthogonal system of simplors has the same density. This same density is called the multiplicity of $\mathfrak{R}$.

Theorem 42.4. *Let a closed ring $\mathfrak{R}$ of uniform multiplicity be isomorphic to a closed ring of projection operators $\mathfrak{T}$ by a correspondence $P^{\mathfrak{T}} \in \mathfrak{T}$ $(P \in \mathfrak{R})$. In order that we can find an isometric operator U with proper domain $C_{\mathfrak{R}}R$ such that*

$$P^{\mathfrak{T}} = U P U^* \qquad \text{for every } P \in \mathfrak{R},$$

it is necessary and sufficient that $\mathfrak{Y}$ also is of uniform multiplicity and the multiplicity of $\mathfrak{Y}$ coincides with that of $\mathfrak{R}$.

Proof. We suppose firstly that $\mathfrak{Y}$ is of uniform multiplicity and has the same multiplicity with $\mathfrak{R}$. We can find by assumption a complete orthogonal system of simplors R_λ $(\lambda \in \Lambda)$ of $\mathfrak{R}$ and S_λ $(\lambda \in \Lambda)$ of $\mathfrak{Y}$. Since $R_\lambda \mathfrak{R}$ is isomorphic to $\mathfrak{R}$ and $S_\lambda \mathfrak{Y}$ is isomorphic to $\mathfrak{Y}$, we see by assumption that $R_\lambda \mathfrak{R}$ is isomorphic to $S_\lambda \mathfrak{Y}$ by the correspondence

$$S_\lambda P^{\mathfrak{Y}} \in S_\lambda \mathfrak{Y} \qquad (R_\lambda P \in R_\lambda \mathfrak{R}).$$

Since both $R_\lambda \mathfrak{R}$ and $S_\lambda \mathfrak{Y}$ are simple, we can find by Theorem 41.4 an isometric operator U_λ with proper domain $R_\lambda R$ such that

$$S_\lambda P^{\mathfrak{Y}} = U_\lambda R_\lambda P U_\lambda^* \qquad \text{for every } P \in \mathfrak{R}.$$

For such U_λ $(\lambda \in \Lambda)$ we have by Theorem 38.1

$$S_\lambda R = U_\lambda R_\lambda U_\lambda^* R = U_\lambda R.$$

Thus putting

$$U x = \sum_{\lambda \in \Lambda} U_\lambda x \qquad \text{for every } x \in R,$$

we obtain an isometric operator U with proper domain

$$(\bigcup_{\lambda \in \Lambda} R_\lambda) R = C_{\mathfrak{R}} R,$$

and we have

$$U R_\lambda = S_\lambda U = U_\lambda \qquad \text{for every } \lambda \in \Lambda.$$

Therefore we have for every $P \in \mathfrak{R}$ and $x \in R$

$$P^{\mathfrak{Y}} x = \sum_{\lambda \in \Lambda} S_\lambda P^{\mathfrak{Y}} x = \sum_{\lambda \in \Lambda} U_\lambda R_\lambda P U_\lambda^* x$$
$$= \sum_{\lambda \in \Lambda} U R_\lambda P U^* x = U P U^* x,$$

because $\sum_{\lambda \in \Lambda} R_\lambda P U^* x = P U^* x$.

Conversely let U be an isometric operator with proper domain D such that

$$C_{\mathfrak{R}} \leqq P_D, \quad P^{\mathfrak{Y}} = U P U^* \qquad \text{for every } P \in \mathfrak{R}.$$

For a complete orthogonal system of simplors R_λ $(\lambda \in \Lambda)$ of $\mathfrak{R}$, we see by Theorem 41.4 that $R_\lambda^{\mathfrak{Y}} \mathfrak{Y}$ also is a simple ring for every $\lambda \in \Lambda$. Furthermore we have for $\lambda \neq \gamma$

$$R_\lambda^{\mathfrak{Y}} R_\gamma^{\mathfrak{Y}} = U (R_\lambda R_\gamma) U^* = 0,$$
$$\bigcup_{\lambda \in \Lambda} U R_\lambda U^* = C_{\mathfrak{R}}^{\mathfrak{Y}} = C_{\mathfrak{Y}}$$

by §41(2), and

$$P^{\mathfrak{Y}} U R_\lambda U^* = 0 \quad \text{if and only if } P R_\lambda = 0.$$

Thus $U R_\lambda U^*$ $(\lambda \in \Lambda)$ is a complete orthogonal system of simplors of $\mathcal{Y}$. Therefore $\mathcal{Y}$ is of uniform multiplicity and has the same multiplicity with $\mathcal{R}$.

§43 Multiplicity Resolutions

Let $\mathcal{R}$ be a closed ring of projection operators. Since

$$\mathcal{C}_{\mathcal{C}_\mathcal{R}} = \mathcal{C}_\mathcal{R},$$

we see by Theorem 42.2 that there exists a simplor of $\mathcal{R}$. By virtue of Maximal Theorem, we can find hence a maximal system of orthogonal simplors S_λ $(\lambda \in \Lambda)$ of $\mathcal{R}$. Then putting

$$Q = \bigcup_{\lambda \in \Lambda} S_\lambda,$$

we have obviously $Q \in \mathcal{C}_\mathcal{R} \overline{\mathcal{R}}$ for the commutor $\overline{\mathcal{R}}$ of $\mathcal{R}$. If $Q = \mathcal{C}_\mathcal{R}$, then $\mathcal{R}$ is of uniform multiplicity by definition. If $Q \neq \mathcal{C}_\mathcal{R}$, then we have

$$\mathcal{C}_{\mathcal{C}_\mathcal{R} - Q} \neq \mathcal{C}_\mathcal{R}.$$

Because if $\mathcal{C}_{\mathcal{C}_\mathcal{R} - Q} = \mathcal{C}_\mathcal{R}$, then we can find by Theorem 42.2 a simplor $S \leqq \mathcal{C}_\mathcal{R} - Q$ of $\mathcal{R}$ and we have obviously $S_\lambda S = 0$ for every $\lambda \in \Lambda$, contradicting the maximality of the system S_λ $(\lambda \in \Lambda)$.

Therefore putting

$$P = \mathcal{C}_\mathcal{R} - \mathcal{C}_{\mathcal{C}_\mathcal{R} - Q},$$

we have $0 \neq P \in \mathcal{R}$ and by the formula §14(16)

$$\mathcal{C}_{P(\mathcal{C}_\mathcal{R} - Q)} = P\,\mathcal{C}_{\mathcal{C}_\mathcal{R} - Q} = 0.$$

This relation yields

$$P = PQ = \bigcup_{\lambda \in \Lambda} P S_\lambda.$$

Furthermore we see by Theorem 20.6 that $P S_\lambda \mathcal{R}$ is a simple ring. On the other hand we have by the formulas §20(2) and §14(16)

$$\mathcal{C}^{P\mathcal{R}}_{P S_\lambda} = P\,\mathcal{C}_{P S_\lambda} = P\,\mathcal{C}_{S_\lambda} = P\,\mathcal{C}_\mathcal{R} = P.$$

Therefore $P S_\lambda$ $(\lambda \in \Lambda)$ is a complete orthogonal system of simplors of the subring $P\mathcal{R}$, and hence the subring $P\mathcal{R}$ is of uniform multiplicity.

Thus we can find $0 \neq P \in \mathcal{R}$ for which the subring $P\mathcal{R}$ is of uniform multiplicity. By virtue of Maximal Theorem, we can find hence a maximal system $0 \neq P_\lambda \in \mathcal{R}$ $(\lambda \in \Lambda)$ such that $P_\lambda P_\rho = 0$ for every $\lambda \neq \rho$, $\bigcup_{\lambda \in \Lambda} P_\lambda = \mathcal{C}_\mathcal{R}$, and $P_\lambda \mathcal{R}$ is of uniform multiplicity for

every $\lambda \in \Lambda$.

For a system $0 \neq P_\gamma \in \mathcal{R}$ $(\gamma \in \Gamma)$, if $P_\lambda P_\rho = 0$ for $\lambda \neq \rho$, and $P_\gamma \mathcal{R}$ is of uniform multiplicity with the same multiplicity for every $\gamma \in \Gamma$, then the subring $(\bigcup_{\gamma \in \Gamma} P_\gamma)\mathcal{R}$ also is of uniform multiplicity with the same multiplicity. In fact, for a complete orthogonal system of simplors $S_{\gamma,\lambda}$ $(\lambda \in \Lambda)$ of $P_\gamma \mathcal{R}$, putting

$$S_\lambda = \bigcup_{\gamma \in \Gamma} S_{\gamma,\lambda} \qquad (\lambda \in \Lambda),$$

we have

$$S_{\gamma,\lambda} = P_\gamma S_\lambda \qquad (\gamma \in \Gamma,\ \lambda \in \Lambda),$$

$$\bigcup_{\lambda \in \Lambda} S_\lambda = \bigcup_{\gamma \in \Gamma} \bigcup_{\lambda \in \Lambda} S_{\gamma,\lambda} = \bigcup_{\gamma \in \Gamma} P_\gamma,$$

and for $\lambda \neq \rho$

$$S_\lambda S_\rho = (\bigcup_{\gamma \in \Gamma} S_{\gamma,\lambda})(\bigcup_{\gamma \in \Gamma} S_{\gamma,\rho}) = \bigcup_{\gamma,\gamma' \in \Gamma} S_{\gamma,\lambda} S_{\gamma',\rho} = 0.$$

For a projection operator Q, if

$$Q(S_\lambda P) = (S_\lambda P)Q \neq 0 \qquad \text{for } S_\lambda P \neq 0,\ P \in \mathcal{R},$$

then for each $\gamma \in \Gamma$ we have naturally

$$Q(S_{\gamma,\lambda} P) = (S_{\gamma,\lambda} P)Q \neq 0 \qquad \text{for } S_{\gamma,\lambda} P \neq 0,\ P \in \mathcal{R}.$$

Thus we obtain by Theorem 20.4 $Q \geqq S_{\gamma,\lambda}$ for every $\gamma \in \Gamma$, and hence $Q \geqq \bigcup_{\gamma \in \Gamma} S_{\gamma,\lambda} = S_\lambda$. Consequently $S_\lambda \mathcal{R}$ is simple by Theorem 20.4. Therefore S_λ $(\lambda \in \Lambda)$ is a complete orthogonal system of simplors of the subring $(\bigcup_{\gamma \in \Gamma} P_\gamma)\mathcal{R}$.

Now we conclude hence that there exists a system $0 \neq P_\lambda \in \mathcal{R}$ $(\lambda \in \Lambda)$ such that $P_\lambda P_\rho = 0$ for $\lambda \neq \rho$, $\bigcup_{\lambda \in \Lambda} P_\lambda = C_{\mathcal{R}}$, $P_\lambda \mathcal{R}$ is of uniform multiplicity for every $\lambda \in \Lambda$, and the multiplicity of $P_\lambda \mathcal{R}$ is different from that of $P_\rho \mathcal{R}$ for $\lambda \neq \rho$. Such a partition P_λ $(\lambda \in \Lambda)$ of $C_{\mathcal{R}}$ is determined uniquely. Because for some $0 \neq P \in \mathcal{R}$, if $P\mathcal{R}$ is of uniform multiplicity and $0 \neq Q \in \mathcal{R}$, $Q \leqq P$, then for a complete orthogonal system of simplors S_λ $(\lambda \in \Lambda)$ of $P\mathcal{R}$ we see easily by Theorem 20.6 that $S_\lambda Q$ $(\lambda \in \Lambda)$ is a complete orthogonal system of simplors of the subring $Q\mathcal{R}$. Thus $Q\mathcal{R}$ also is of uniform multiplicity and the multiplicity of $Q\mathcal{R}$ coincides with that of $P\mathcal{R}$. Therefore we can now state

Theorem 43.1. *For a closed ring of projection operators $\mathcal{R}$ there exists uniquely a system $0 \neq P_\lambda \in \mathcal{R}$ $(\lambda \in \Lambda)$ such that*

$$P_\lambda P_\rho = 0 \quad \text{for } \lambda \neq \rho, \quad \bigcup_{\lambda \in \Lambda} P_\lambda = C_{\mathcal{R}},$$

$P_\lambda \mathcal{R}$ is of uniform multiplicity for every $\lambda \in \Lambda$, and the multiplicity of $P_\lambda \mathcal{R}$ is different from that of $P_\rho \mathcal{R}$ for $\lambda \neq \rho$.

This uniquely determined partition $P_\lambda \in \mathcal{R}$ $(\lambda \in \Lambda)$ of $C_{\mathcal{R}}$ is called the multiplicity resolution of $\mathcal{R}$, and every P_λ is called a multiplicity term of $\mathcal{R}$ for the multiplicity of $P_\lambda \mathcal{R}$.

Theorem 43.2. Let a closed ring $\mathcal{R}$ be isomorphic to a closed ring $\mathcal{T}$ by a correspondence $P \in \mathcal{R} \to P^{\mathcal{T}} \in \mathcal{T}$. In order that we can find an isometric operator U with proper domain $C_{\mathcal{R}} R$ such that

$$P^{\mathcal{T}} = U P U^* \quad \text{for every } P \in \mathcal{R},$$

it is necessary and sufficient that for every multiplicity term P of $\mathcal{R}$, $P^{\mathcal{T}}$ also is a multiplicity term of $\mathcal{T}$ with the same multiplicity.

Proof. We suppose firstly that for every multiplicity term P of $\mathcal{R}$, $P^{\mathcal{T}}$ also is a multiplicity term of $\mathcal{T}$ with the same multiplicity. For the multiplicity resolution P_λ $(\lambda \in \Lambda)$ of $\mathcal{R}$, we see by assumption that $P_\lambda^{\mathcal{T}}$ $(\lambda \in \Lambda)$ also is the multiplicity resolution of $\mathcal{T}$ and the multiplicity of $P_\lambda^{\mathcal{T}} \mathcal{T}$ coincides with that of $P_\lambda \mathcal{R}$ for every $\lambda \in \Lambda$. Thus we can find by Theorem 42.4 an isometric operator U_λ with proper domain $P_\lambda R$ such that

$$P_\lambda^{\mathcal{T}} P^{\mathcal{T}} = U_\lambda P_\lambda P U_\lambda^* \quad \text{for every } P \in \mathcal{R}.$$

For such U_λ $(\lambda \in \Lambda)$, putting

$$U x = \sum_{\lambda \in \Lambda} U_\lambda x \quad \text{for every } x \in R,$$

we obtain an isometric operator U with proper domain $C_{\mathcal{R}} R$, because

$$U_\lambda = P_\lambda^{\mathcal{T}} U_\lambda \quad \text{for every } \lambda \in \Lambda,$$

$$\sum_{\lambda \in \Lambda} \| U_\lambda x \|^2 = \sum_{\lambda \in \Lambda} \| P_\lambda x \|^2 = \| \sum_{\lambda \in \Lambda} P_\lambda x \|^2 = \| C_{\mathcal{R}} x \|^2.$$

Furthermore we have for every $\lambda \in \Lambda$

$$P_\lambda^{\mathcal{T}} U = U P_\lambda = U_\lambda,$$

and hence for every $P \in \mathcal{R}$

$$P_\lambda^{\mathcal{T}} U P U^* = U P_\lambda P U^* = U_\lambda P_\lambda P U_\lambda^* = P_\lambda^{\mathcal{T}} P^{\mathcal{T}}.$$

Thus we have for every $x \in R$

$$U P U^* x = \sum_{\lambda \in \Lambda} P_\lambda^{\mathcal{T}} U P U^* x = \sum_{\lambda \in \Lambda} P_\lambda^{\mathcal{T}} P^{\mathcal{T}} x = P^{\mathcal{T}} x,$$

because $\bigcup_{\lambda \in \Lambda} P_\lambda^{\mathcal{T}} = C_{\mathcal{T}}$.

Conversely if $\mathcal{R}$ is isomorphic to $\mathcal{T}$ by a correspondence

$$U P U^* \in \mathcal{T} \qquad (P \in \mathcal{R})$$

for an isometric operator with proper domain D such that $P_D \geqq C_{\mathcal{R}}$, then we see by Theorem 42.2 that for the multiplicity resolution P_λ $(\lambda \in \Lambda)$ of $\mathcal{R}$, $U P_\lambda U^*$ $(\lambda \in \Lambda)$ also is the multiplicity resolution of $\mathcal{T}$ and the multiplicity of $U P_\lambda U^* \mathcal{T}$ coincides with that of $P_\lambda \mathcal{R}$ for every $\lambda \in \Lambda$.

Theorem 43.3. *Let the dimension $\mathfrak{l}$ of a Hilbert space R be infinite and $\mathcal{R}$ a closed ring of projection operators in R. For a system $0 \neq P_\lambda \in \mathcal{R}$ $(\lambda \in \Lambda)$ subject to the condition:*

$$P_\lambda P_\rho = 0 \quad \text{for } \lambda \neq \rho, \qquad \bigcup_{\lambda \in \Lambda} P_\lambda = C_{\mathcal{R}},$$

and for a system of different cardinal numbers $\mathfrak{l}_\lambda \leqq \mathfrak{l}$ $(\lambda \in \Lambda)$, we can find a closed ring of projection operators $\mathcal{T}$ such that $\mathcal{R}$ is isomorphic to $\mathcal{T}$ by a correspondence $P \to P^{\mathcal{T}} \in \mathcal{T}$ $(P \in \mathcal{R})$ and $P_\lambda^{\mathcal{T}}$ is a multiplicity term of $\mathcal{T}$ for the multiplicity $\mathfrak{l}_\lambda$.

Proof. Since $\mathfrak{l}_\lambda \leqq \mathfrak{l}$ $(\lambda \in \Lambda)$ are mutually different, the density of Λ is not greater than $\mathfrak{l}$. Let Γ_λ be a set with density $\mathfrak{l}_\lambda$ for every $\lambda \in \Lambda$. Then the density of $\sum_{\lambda \in \Lambda} \Gamma_\lambda$ also is not greater than $\mathfrak{l}$. Thus we can find a system of closed linear manifolds $M_{\lambda,\gamma}$ $(\gamma \in \Gamma_\lambda, \lambda \in \Lambda)$ such that the dimension of $M_{\lambda,\gamma}$ is $\mathfrak{l}$ for every $\gamma \in \Gamma_\lambda$, $\lambda \in \Lambda$, and

$$M_{\lambda,\gamma} M_{\lambda',\gamma'} = \{0\} \qquad \text{for } \lambda \neq \lambda' \text{ or } \gamma \neq \gamma'.$$

For a simplor S of $\mathcal{R}$, $\mathcal{R}$ is isomorphic to $S\mathcal{R}$ by definition. Thus we can assume without loss of generality that $\mathcal{R}$ is simple. Since every $M_{\lambda,\gamma}$ has the same dimension with R, there exists an isometric operator $U_{\lambda,\gamma}$ with proper domain R and range $M_{\lambda,\gamma}$. For such $U_{\lambda,\gamma}$ $(\gamma \in \Gamma_\lambda, \lambda \in \Lambda)$, putting

$$P^{\mathcal{T}} = \bigcup_{\lambda \in \Lambda} \bigcup_{\gamma \in \Gamma_\lambda} U_{\lambda,\gamma} P_\lambda P U_{\lambda,\gamma}^* \qquad \text{for } P \in \mathcal{R},$$

we have for every $\lambda \in \Lambda$ and $\gamma \in \Gamma_\lambda$

$$P_\lambda^{\mathcal{T}} = \bigcup_{\gamma \in \Gamma_\lambda} U_{\lambda,\gamma} P_\lambda U_{\lambda,\gamma}^*,$$

$$U_{\lambda,\gamma} P_\lambda U_{\lambda,\gamma}^* P^{\mathcal{T}} = U_{\lambda,\gamma} P_\lambda P U_{\lambda,\gamma}^* \quad \text{for every } P \in \mathcal{R},$$

$$(U_{\lambda,\gamma} P_\lambda U_{\lambda,\gamma}^*)(U_{\lambda,\gamma'} P_\lambda U_{\lambda,\gamma'}^*) = 0 \quad \text{for } \gamma \neq \gamma',$$

$$\bigcup_{\lambda\in\Lambda} P_\lambda^{\mathcal{T}} = C_{\mathcal{R}}^{\mathcal{T}}.$$

Thus putting

$$\mathcal{T} = \{ P^{\mathcal{T}} : \ P \in \mathcal{R} \},$$

we see by Theorem 41.4 that $U_{\lambda,\gamma} P_\lambda U_{\lambda,\gamma}^* \mathcal{T}$ is a simple ring, because $\mathcal{R}$ is simple by assumption and hence $P_\lambda \mathcal{R}$ also is a simple ring by Theorem 20.6. Furthermore $\mathcal{T}$ is a closed ring. In fact we have by Theorem 13.9

$$(C_{\mathcal{R}} - P)^{\mathcal{T}} = C_{\mathcal{R}}^{\mathcal{T}} - P^{\mathcal{T}} \quad \text{for every } P \in \mathcal{R},$$

and by §41(2) for every system $Q_\delta \in \mathcal{R}$ $(\delta \in \Delta)$

$$(\bigcup_{\delta\in\Delta} Q_\delta)^{\mathcal{T}} = \bigcup_{\delta\in\Delta} Q_\delta^{\mathcal{T}}.$$

Therefore we conclude easily by definition that $P_\lambda^{\mathcal{T}}$ $(\lambda \in \Lambda)$ is the multiplicity resolution of $\mathcal{T}$ and the multiplicity of $P_\lambda^{\mathcal{T}}\mathcal{T}$ is ν_λ for every $\lambda \in \Lambda$. Furthermore it is obvious that $\mathcal{R}$ is isomorphic to $\mathcal{T}$ by the correspondence $P^{\mathcal{T}} \in \mathcal{T}$ $(P \in \mathcal{R})$.

§44 Multiplicity Functions

Let $\mathcal{R}$ be a closed ring of projection operators with multiplicity resolution $R_\gamma \in \mathcal{R}$ $(\gamma \in \Gamma)$ and ν_γ the multiplicity of $R_\gamma \mathcal{R}$ for every $\gamma \in \Gamma$. For each $P \in \mathcal{R}$ we define the minimal multiplicity $m(P)$ to mean

$$(1) \qquad m(P) = \operatorname*{Min}_{PR_\gamma \neq 0} \nu_\gamma \quad \text{for } 0 \neq P \in \mathcal{R},$$

and $m(0) = 0$. If we need indicate $\mathcal{R}$, then we denote by $m_{\mathcal{R}}(P)$ the minimal multiplicity of P.

Concerning minimal multiplicities, we have obviously by definition

$$(2) \qquad m(R_\gamma) = \nu_\gamma \qquad (\gamma \in \Gamma),$$

$$(3) \qquad m(P) \leqq m(Q) \quad \text{for } P \geqq Q \neq 0,$$

$$(4) \qquad m(\bigcup_{\lambda\in\Lambda} P_\lambda) = \operatorname*{Min}_{R_\lambda \neq 0, \lambda\in\Lambda} m(P_\lambda).$$

Theorem 14.1. *For a projection operator $0 \neq P_0 \in \mathcal{R}$, the subring $P_0\mathcal{R}$ is of uniform multiplicity with multiplicity ν, if and only if the minimal multiplicity of P coincides with ν for every $0 \neq P \in P_0\mathcal{R}$, that is, $m(P) = \nu$ for $0 \neq P \in P_0\mathcal{R}$.*

Proof. If $P_0\mathcal{R}$ is of uniform multiplicity with multiplicity ν, then there exists a multiplicity term P_1 of $\mathcal{R}$ for the multiplicity ν and $P_0 \leqq P_1$. Thus we have by definition

$$m(P) = \nu \qquad \text{for } 0 \neq P \in P_0\mathcal{R}.$$

Conversely if $m(P) = \nu$ for $0 \neq P \in P_0\mathcal{R}$, then for the multiplicity resolution $R_\gamma \in \mathcal{R}$ ($\gamma \in \Gamma$) and for the multiplicity ν_γ of $R_\gamma\mathcal{R}$ we have by definition

$$P_0 R_\gamma = 0 \qquad \text{for } \nu_\gamma \neq \nu.$$

Thus there exists $\gamma \in \Gamma$ for which $P_0 \leqq R_\gamma$, $\nu_\gamma = \nu$, and hence the subring $P_0\mathcal{R}$ is of uniform multiplicity with multiplicity ν.

Recalling Theorem 43.2, we obtain by this Theorem 44.1

Theorem 44.2. *Let a closed ring $\mathcal{R}$ be isomorphic to a closed ring $\mathcal{T}$ by a correspondence $P \to P^{\mathcal{T}} \in \mathcal{T}$ ($P \in \mathcal{R}$). In order that there exists an isometric operator U with proper domain $\mathfrak{C}_{\mathcal{R}}R$ such that*

$$P^{\mathcal{T}} = UPU^* \qquad \text{for every } P \in \mathcal{R},$$

it is necessary and sufficient that

$$m_{\mathcal{T}}(P^{\mathcal{T}}) = m_{\mathcal{R}}(P) \qquad \text{for every } P \in \mathcal{R},$$

denoting by $m_{\mathcal{R}}$ the minimal multiplicity of $\mathcal{R}$ and by $m_{\mathcal{T}}$ that of $\mathcal{T}$.

Let $\mathcal{R}$ be a closed ring of projection operators and $m(P)$ ($P \in \mathcal{R}$) the minimal multiplicity of $\mathcal{R}$. For the proper space $\mathfrak{L}_{\mathcal{R}}$ of $\mathcal{R}$, putting

$$(5) \qquad m(\mathfrak{P}) = \sup_{P \in \mathfrak{P}} m(P) \qquad \text{for } \mathfrak{P} \in \mathfrak{L}_{\mathcal{R}},$$

we obtain a cardinal valued function m on $\mathfrak{L}_{\mathcal{R}}$. This function m on $\mathfrak{L}_{\mathcal{R}}$ is called the multiplicity function of $\mathcal{R}$ and the function value $m(\mathfrak{P})$ the multiplicity of $\mathcal{R}$ at $\mathfrak{P}$.

Concerning the multiplicity function m, we have

$$(6) \qquad m(P) = \operatorname*{Min}_{\mathfrak{P} \ni P} m(\mathfrak{P}) \qquad \text{for } 0 \neq P \in \mathcal{R}.$$

In fact it is obvious by the definition (5) that $m(P) \leqq \operatorname*{Min}_{\mathfrak{P} \ni P} m(\mathfrak{P})$ for $0 \neq P \in \mathcal{R}$. On the other hand, we can find by the definition (1) a multiplicity term R_γ of $\mathcal{R}$ such that $PR_\gamma \neq 0$ and $m(P) = \nu_\gamma$ for the multiplicity ν_γ of $R_\gamma\mathcal{R}$. Then for each $\mathfrak{P} \ni PR_\gamma$

we conclude by Theorem 44.1 that $m(\mathfrak{P}) = \aleph_\gamma$. Therefore we obtain (6).

Theorem 44.3. The multiplicity function m of $\mathcal{R}$ is a continuous function on $\mathfrak{S}_{\mathcal{R}}$ and

$$\{\mathfrak{P} : m(\mathfrak{P}) > \aleph\}$$

is open and closed for every cardinal number $\aleph$.

Proof. If $m(\mathfrak{P}_0) > \aleph$, then there exists by the definition (5) $P \in \mathfrak{P}_0$ for which $m(P) > \aleph$. Furthermore we have by (5)

$$m(\mathfrak{P}) \geqq m(P) \qquad \text{for every } \mathfrak{P} \in \mathfrak{U}_P.$$

Therefore $\{\mathfrak{P} : m(\mathfrak{P}) > \aleph\}$ is open.

If $\{\mathfrak{P} : m(\mathfrak{P}) > \aleph\} \neq 0$, then we can find a system of projection operators $0 \neq P_\lambda \in \mathcal{R}$ $(\lambda \in \Lambda)$ such that

$$\{\mathfrak{P} : m(\mathfrak{P}) > \aleph\} = \sum_{\lambda \in \Lambda} \mathfrak{U}_{P_\lambda},$$

because $\{\mathfrak{P} : m(\mathfrak{P}) > \aleph\}$ is open, as proved just above. Thus we have by Theorem 15.7

$$\{\mathfrak{P} : m(\mathfrak{P}) > \aleph\}^- = \mathfrak{U}_{\bigcup_{\lambda \in \Lambda} P_\lambda}.$$

On the other hand we have by the formula (6)

$$m(P_\lambda) > \aleph \qquad \text{for every } \lambda \in \Lambda,$$

and hence by (4)

$$m\left(\bigcup_{\lambda \in \Lambda} P_\lambda\right) > \aleph.$$

This relation yields by the definition (5)

$$m(\mathfrak{P}) > \aleph \qquad \text{for every } \mathfrak{P} \in \mathfrak{U}_{\bigcup_{\lambda \in \Lambda} P_\lambda}.$$

Therefore we have

$$\{\mathfrak{P} : m(\mathfrak{P}) > \aleph\}^- = \{\mathfrak{P} : m(\mathfrak{P}) > \aleph\},$$

and hence $\{\mathfrak{P} : m(\mathfrak{P}) > \aleph\}$ is closed.

Since

$$\{\mathfrak{P} : m(\mathfrak{P}) \geqq \aleph\} = \prod_{\mathfrak{c} < \aleph} \{\mathfrak{P} : m(\mathfrak{P}) > \mathfrak{c}\},$$

we conclude hence that $\{\mathfrak{P} : m(\mathfrak{P}) \geqq \aleph\}$ is closed. Therefore m is continuous.

Theorem 44.4. For a multiplicity term R_γ of $\mathcal{R}$ with multiplicity $\aleph_\gamma$ we have

$$\mathfrak{U}_{R_\gamma} = \{\mathfrak{P} : m(\mathfrak{P}) = \aleph_\gamma\}^\circ$$

for the multiplicity function m of $\mathcal{R}$.

Proof. Since we have by the definition (5)

$$m(\mathfrak{F}) = \nu_\gamma \qquad \text{for every } \mathfrak{F} \in \mathcal{O}_{R_\gamma},$$

we obtain immediately

$$\mathcal{O}_{R_\gamma} \subset \{\mathfrak{F} : m(\mathfrak{F}) = \nu_\gamma\}^\circ.$$

Since $\{\mathfrak{F} : m(\mathfrak{F}) = \nu_\gamma\}$ is closed by Theorem 44.3, we can find by Theorems 15.9 and 15.6 $P_0 \in \mathcal{R}$ for which

$$\mathcal{O}_{P_0} = \{\mathfrak{F} : m(\mathfrak{F}) = \nu_\gamma\}^\circ.$$

For such P_0 we have obviously $R_\gamma \leqq P_0$, and the subring $P_0\mathcal{R}$ is of uniform multiplicity by Theorem 44.1. Therefore we obtain

$$R_\gamma = P_0.$$

Theorem 44.5. For two closed rings of projection operators $\mathcal{R}$ and $\mathcal{T}$ there exists an isometric operator U with proper domain $C_{\mathcal{R}}R$ such that $\mathcal{R}$ is isomorphic to $\mathcal{T}$ by the correspondence

$$U P U^* \in \mathcal{T} \qquad (P \in \mathcal{R}),$$

if and only if the proper space $\mathcal{L}_{\mathcal{R}}$ of $\mathcal{R}$ is homeomorphic to the proper space $\mathcal{L}_{\mathcal{T}}$ of $\mathcal{T}$ by a correspondence $\mathfrak{F}^{\mathcal{T}} \in \mathcal{L}_{\mathcal{T}}$ $(\mathfrak{F} \in \mathcal{L}_{\mathcal{R}})$ such that

$$m_{\mathcal{T}}(\mathfrak{F}^{\mathcal{T}}) = m_{\mathcal{R}}(\mathfrak{F}) \qquad \text{for every } \mathfrak{F} \in \mathcal{L}_{\mathcal{R}},$$

denoting by $m_{\mathcal{R}}$ the multiplicity function of $\mathcal{R}$ and by $m_{\mathcal{T}}$ that of $\mathcal{T}$.

Proof. If $\mathcal{L}_{\mathcal{R}}$ is homeomorphic to $\mathcal{L}_{\mathcal{T}}$ by a correspondence $\mathfrak{F}^{\mathcal{T}} \in \mathcal{L}_{\mathcal{T}}$ $(\mathfrak{F} \in \mathcal{L}_{\mathcal{R}})$, then there exists by Theorem 41.2 an isomorphism $P^{\mathcal{T}} \in \mathcal{T}$ $(P \in \mathcal{R})$ such that

$$\mathcal{O}_{P^{\mathcal{T}}} = \{\mathfrak{F}^{\mathcal{T}} : \mathfrak{F} \in \mathcal{O}_P\} \qquad \text{for } P \in \mathcal{R}.$$

Furthermore if $m_{\mathcal{T}}(\mathfrak{F}^{\mathcal{T}}) = m_{\mathcal{R}}(\mathfrak{F})$ for every $\mathfrak{F} \in \mathcal{L}_{\mathcal{R}}$, then we have by the formula (6) for every $0 \neq P \in \mathcal{R}$

$$m_{\mathcal{T}}(P^{\mathcal{T}}) = \operatorname*{Min}_{\mathfrak{F}^{\mathcal{T}} \ni P^{\mathcal{T}}} m_{\mathcal{T}}(\mathfrak{F}^{\mathcal{T}}) = \operatorname*{Min}_{\mathfrak{F} \ni P} m_{\mathcal{R}}(\mathfrak{F}) = m_{\mathcal{R}}(P).$$

Therefore there exists by Theorem 44.2 an isometric operator U with proper domain $C_{\mathcal{R}}R$ such that

$$P^{\mathcal{T}} = U P U^* \qquad \text{for every } P \in \mathcal{R}.$$

Conversely if $\mathcal{R}$ is isomorphic to $\mathfrak{r}$ by a correspondence

$$U P U^* \in \mathfrak{r} \qquad (P \in \mathcal{R})$$

for an isometric operator U with proper domain $C_{\mathcal{R}} R$, then we have by Theorem 44.2

$$m_{\mathfrak{r}}(U P U^*) = m_{\mathcal{R}}(P) \qquad \text{for every } P \in \mathcal{R}.$$

On the other hand we see by Theorem 41.1 that $\mathcal{L}_{\mathcal{R}}$ is homeomorphic to $\mathcal{L}_{\mathfrak{r}}$ by a correspondence $\mathfrak{F}^{\mathfrak{r}} \in \mathcal{L}_{\mathfrak{r}}$ $(\mathfrak{F} \in \mathcal{L}_{\mathcal{R}})$ such that we have $\mathfrak{F}^{\mathfrak{r}} \ni U P U^*$ if and only if $\mathfrak{F} \ni P$. Thus we have by (6) for every $\mathfrak{F} \in \mathcal{L}_{\mathcal{R}}$

$$m_{\mathfrak{r}}(\mathfrak{F}^{\mathfrak{r}}) = \sup_{\mathfrak{F}^{\mathfrak{r}} \ni U P U^*} m_{\mathfrak{r}}(U P U^*) = \sup_{\mathfrak{F} \ni P} m_{\mathcal{R}}(P) = m_{\mathcal{R}}(\mathfrak{F}).$$

Theorem 44.6. *Let the dimension $\mathfrak{n}$ of a Hilbert space R be infinite and $\mathcal{R}$ a closed ring of projection operators in R. For a cardinal valued continuous function $\mathfrak{c}(\mathfrak{F}) \leqq \mathfrak{n}$ $(\mathfrak{F} \in \mathcal{L}_{\mathcal{R}})$ on the proper space $\mathcal{L}_{\mathcal{R}}$ of $\mathcal{R}$ we can find a closed ring $\mathfrak{r}$ such that $\mathcal{L}_{\mathcal{R}}$ is homeomorphic to the proper space $\mathcal{L}_{\mathfrak{r}}$ of $\mathfrak{r}$ by a correspondence*

$$\mathfrak{F}^{\mathfrak{r}} \in \mathcal{L}_{\mathfrak{r}} \qquad (\mathfrak{F} \in \mathcal{L}_{\mathcal{R}})$$

such that for the multiplicity function $m_{\mathfrak{r}}$ of $\mathfrak{r}$ we have

$$m_{\mathfrak{r}}(\mathfrak{F}^{\mathfrak{r}}) = \mathfrak{c}(\mathfrak{F}) \qquad \text{for every } \mathfrak{F} \in \mathcal{L}_{\mathcal{R}}.$$

Proof. Since $\mathfrak{c}(\mathfrak{F})$ $(\mathfrak{F} \in \mathcal{L}_{\mathcal{R}})$ is continuous by assumption, the point set

$$\{\mathfrak{F} : \mathfrak{c}(\mathfrak{F}) = \mathfrak{a}\}^\circ$$

is open and closed by Theorem 15.9 for every cardinal number $\mathfrak{a}$, and hence there exists uniquely $P_{\mathfrak{a}} \in \mathcal{R}$ for which

$$U_{P_{\mathfrak{a}}} = \{\mathfrak{F} : \mathfrak{c}(\mathfrak{F}) = \mathfrak{a}\}^\circ.$$

Thus we can find a system of different cardinal numbers $\mathfrak{n}_\lambda \leqq \mathfrak{n}$ and projection operators $P_{\mathfrak{n}_\lambda} \in \mathcal{R}$ $(\lambda \in \Lambda)$ such that

$$U_{P_{\mathfrak{n}_\lambda}} = \{\mathfrak{F} : \mathfrak{c}(\mathfrak{F}) = \mathfrak{n}_\lambda\}^\circ,$$

$$P_{\mathfrak{n}_\lambda} \neq 0 \qquad \text{for every } \lambda \in \Lambda,$$

$$\bigcup_{\lambda \in \Lambda} P_{\mathfrak{n}_\lambda} = C_{\mathcal{R}}.$$

Then, since $U_{P_{\mathfrak{n}_\lambda}} U_{P_{\mathfrak{n}_\rho}} = 0$ for $\lambda \neq \rho$, we can find by Theorem 43.3 a closed ring $\mathfrak{r}$ such that $\mathcal{R}$ is isomorphic to $\mathfrak{r}$ by a correspondence

$P\mathfrak{x}^{\mathfrak{r}} \in \mathfrak{r}$ $(P \in \mathfrak{R})$ and $P_{\mathfrak{x}_\lambda}{}^{\mathfrak{r}}$ is a multiplicity term of $\mathfrak{r}$ with multiplicity $\mathfrak{x}_\lambda$. For such an isomorphism $P\mathfrak{x}^{\mathfrak{r}} \in \mathfrak{r}$ $(P \in \mathfrak{R})$ there exists by Theorem 41.1 a homeomorphic correspondence $\mathfrak{x}^{\mathfrak{r}} \in \mathfrak{X}_{\mathfrak{r}}$ $(\mathfrak{x} \in \mathfrak{X}_{\mathfrak{R}})$ such that

$$U_{P_{\mathfrak{x}_\lambda}{}^{\mathfrak{r}}} = \{\mathfrak{x}^{\mathfrak{r}} : \quad \tau(\mathfrak{x}) = \mathfrak{x}_\lambda\}^\circ \qquad \text{for every } \lambda \in \Lambda,$$

and hence for the multiplicity function $m_{\mathfrak{r}}$ of $\mathfrak{r}$ we obtain by the definition (5)

$$m_{\mathfrak{r}}(\mathfrak{x}^{\mathfrak{r}}) = \tau(\mathfrak{x}) \qquad \text{for } \mathfrak{x} \in \sum_{\lambda \in \Lambda} U_{P_{\mathfrak{x}_\lambda}}.$$

Since τ is continuous over $\mathfrak{X}_{\mathfrak{R}}$ and $\sum_{\lambda \in \Lambda} U_{P_{\mathfrak{x}_\lambda}}$ is dense in $\mathfrak{X}_{\mathfrak{R}}$, we conclude therefore

$$m_{\mathfrak{r}}(\mathfrak{x}^{\mathfrak{r}}) = \tau(\mathfrak{x}) \qquad \text{for every } \mathfrak{x} \in \mathfrak{X}_{\mathfrak{R}}.$$

§45 σ-ideals of Sets

Let $\mathfrak{M}$ be a totally additive set class in an abstract space Ω. A subclass $\mathfrak{x} \subset \mathfrak{M}$ is said to be a σ-ideal in $\mathfrak{M}$, if

1) $\mathfrak{M} \ni \Psi \supset \Phi \in \mathfrak{x}$ implies $\Psi \in \mathfrak{x}$,

2) $\Phi_\nu \in \mathfrak{x}$ $(\nu = 1, 2, \ldots)$ implies $\prod_{\nu=1}^{\infty} \Phi_\nu \in \mathfrak{x}$.

With this definition it is evident that $\Omega \in \mathfrak{x}$ for every σ-ideal $\mathfrak{x}$ and $\mathfrak{M}$ is itself a σ-ideal. Furthermore we see easily by definition that for every sequence of σ-ideals $\mathfrak{x}_\nu$ $(\nu = 1, 2, \ldots)$ the intersection $\prod_{\nu=1}^{\infty} \mathfrak{x}_\nu$ also is a σ-ideal.

For two σ-ideals $\mathfrak{x}$ and $\mathfrak{y}$ in $\mathfrak{M}$ we shall write $\mathfrak{x} \perp \mathfrak{y}$, if we can find $\Phi \in \mathfrak{x}$ and $\Psi \in \mathfrak{y}$ such that $\Phi\Psi = 0$. With this definition we have $\mathfrak{x} \perp \mathfrak{M}$ for every σ-ideal $\mathfrak{x}$, because $0 \in \mathfrak{M}$.

Let $\mathfrak{x}$ be a σ-ideal in $\mathfrak{M}$. For each $\Phi_0 \in \mathfrak{M}$, putting

$$\mathfrak{y} = \{\Phi : \quad \Phi \dot{+} \Phi_0' \in \mathfrak{x}, \quad \Phi \in \mathfrak{M}\},$$

we obtain a σ-ideal $\mathfrak{y}$ in $\mathfrak{M}$. In fact, we have obviously by 1) that $\mathfrak{M} \ni \Psi \supset \Phi \in \mathfrak{y}$ implies $\Psi \in \mathfrak{y}$. If $\Phi_\nu \in \mathfrak{y}$ $(\nu = 1, 2, \ldots)$, then we have by 2)

$$\prod_{\nu=1}^{\infty} \Phi_\nu \dot{+} \Phi_0' = \prod_{\nu=1}^{\infty} (\Phi_\nu \dot{+} \Phi_0') \in \mathfrak{x},$$

and hence $\prod_{\nu=1}^{\infty} \Phi_\nu \in \mathfrak{y}$. This σ-ideal $\mathfrak{y}$ is called a co-ideal of a σ-ideal $\mathfrak{x}$ by Φ_0 and denoted by $\mathfrak{x}^{\Phi_0}$, that is,

(3) $\mathfrak{P}^{\mathfrak{E}_0} = \{\mathfrak{E} : \mathfrak{E} \dot{+} \mathfrak{E}_0' \in \mathfrak{P},\ \mathfrak{E} \in \mathfrak{M}\}$.

Since we have by 1) $\mathfrak{E} \dot{+} \mathfrak{E}_0' \in \mathfrak{P}$ for every $\mathfrak{E} \in \mathfrak{P}$, we obtain by definition (3)

(4) $\mathfrak{P}^{\mathfrak{E}} \supset \mathfrak{P}$ for every $\mathfrak{E} \in \mathfrak{M}$.

If $\mathfrak{E}_0 \in \mathfrak{P}$, then $\mathfrak{E} \dot{+} \mathfrak{E}_0' \in \mathfrak{P}$ implies by 2)

$$\mathfrak{E} \supset \mathfrak{E}_0(\mathfrak{E} \dot{+} \mathfrak{E}_0') \in \mathfrak{P},$$

and hence $\mathfrak{E} \in \mathfrak{P}$ by 1). Thus $\mathfrak{P}^{\mathfrak{E}_0} \subset \mathfrak{P}$ for $\mathfrak{E}_0 \in \mathfrak{P}$. Consequently we obtain by (4)

(5) $\mathfrak{P}^{\mathfrak{E}} = \mathfrak{P}$ for every $\mathfrak{E} \in \mathfrak{P}$.

Since $\mathfrak{E} \dot{+} \mathfrak{E}' = \Omega \in \mathfrak{P}$ for every $\mathfrak{E} \in \mathfrak{M}$, we obtain by (3)

(6) $\mathfrak{E} \in \mathfrak{P}^{\mathfrak{E}}$ for every $\mathfrak{E} \in \mathfrak{M}$.

From this relation (6) we conclude immediately

(7) $\mathfrak{P}^{\mathfrak{E}} \perp \mathfrak{P}^{\mathfrak{F}}$ for $\mathfrak{E}\mathfrak{F} = 0$.

For a sequence $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ we have by the definition (3)

$$\begin{aligned}\mathfrak{P}^{\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu} &= \{\mathfrak{E} : \mathfrak{E} \dot{+} (\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu)' \in \mathfrak{P}\} \\ &= \{\mathfrak{E} : \prod_{\nu=1}^{\infty} (\mathfrak{E} \dot{+} \mathfrak{E}_\nu') \in \mathfrak{P}\} \\ &= \prod_{\nu=1}^{\infty} \{\mathfrak{E} : \mathfrak{E} \dot{+} \mathfrak{E}_\nu' \in \mathfrak{P}\}.\end{aligned}$$

Thus we have

(8) $\mathfrak{P}^{\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu} = \prod_{\nu=1}^{\infty} \mathfrak{P}^{\mathfrak{E}_\nu}$ for $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$.

For every $\mathfrak{E}_1$, $\mathfrak{E}_2 \in \mathfrak{M}$ we have by the definition (3)

$$(\mathfrak{P}^{\mathfrak{E}_1})^{\mathfrak{E}_2} = \{\mathfrak{E} : \mathfrak{E} \dot{+} \mathfrak{E}_2' \in \mathfrak{P}^{\mathfrak{E}_1}\}$$
$$= \{\mathfrak{E} : \mathfrak{E} \dot{+} \mathfrak{E}_2' \dot{+} \mathfrak{E}_1' \in \mathfrak{P}\} = \{\mathfrak{E} : \mathfrak{E} \dot{+} (\mathfrak{E}_1\mathfrak{E}_2)' \in \mathfrak{P}\}.$$

Thus we obtain

(9) $(\mathfrak{P}^{\mathfrak{E}})^{\mathfrak{F}} = \mathfrak{P}^{\mathfrak{E}\mathfrak{F}}$ for every $\mathfrak{E}, \mathfrak{F} \in \mathfrak{M}$.

A σ-ideal $\mathfrak{P}$ is said to be <u>separable</u>, if $\mathfrak{E}_\lambda \mathfrak{E}_\rho = 0$ for $\lambda \neq \rho$, $\mathfrak{E}_\lambda \in \mathfrak{M}$ $(\lambda \in \Lambda)$, implies $\mathfrak{E}_\lambda' \in \mathfrak{P}$ except for at most countable elements $\lambda \in \Lambda$. The totality of separable σ-ideals in $\mathfrak{M}$ is called the <u>property</u> of $\mathfrak{M}$ and denoted by $\mathcal{P}_{\mathfrak{M}}$.

With this definition we have obviously

(10) $\mathfrak{q} \supset \mathfrak{p} \in \mathcal{P}_{\mathfrak{M}}$ implies $\mathfrak{q} \in \mathcal{P}_{\mathfrak{M}}$.

Furthermore we have

Theorem 45.1. If $\mathfrak{q} \supset \mathfrak{p} \in \mathcal{P}_{\mathfrak{M}}$, then $\mathfrak{q}$ is a co-ideal of $\mathfrak{p}$, that is, we can find $\mathfrak{E}_0 \in \mathfrak{M}$, for which $\mathfrak{q} = \mathfrak{p}^{\mathfrak{E}_0}$.

Proof. By virtue of the formula (5), we need only consider the case where $\mathfrak{q} \neq \mathfrak{p}$. By virtue of Maximal Theorem, we can find a maximal system $\mathfrak{E}_\lambda \in \mathfrak{M}$ $(\lambda \in \Lambda)$ subject to the condition:

$$\mathfrak{E}_\lambda \mathfrak{E}_\rho = 0 \qquad \text{for } \lambda \neq \rho,$$

$$\mathfrak{E}_\lambda' \in \mathfrak{q}, \quad \mathfrak{E}_\lambda' \bar{\in} \mathfrak{p} \qquad \text{for every } \lambda \in \Lambda.$$

Such Λ is at most countable, because $\mathfrak{p}$ is separable by assumption. Thus, putting

$$\mathfrak{E}_0 = \prod_{\lambda \in \Lambda} \mathfrak{E}_\lambda',$$

we obtain $\mathfrak{E}_0 \in \mathfrak{q}$ by 2).

If $\mathfrak{E} \dotplus \mathfrak{E}_0' \bar{\in} \mathfrak{p}$ for some $\mathfrak{E} \in \mathfrak{q}$, then we have

$$(\mathfrak{E}'\mathfrak{E}_0)\mathfrak{E}_\lambda \subset \mathfrak{E}'\mathfrak{E}_\lambda'\mathfrak{E}_\lambda = 0 \qquad \text{for every } \lambda \in \Lambda,$$

and $(\mathfrak{E}'\mathfrak{E}_0)' = \mathfrak{E} \dotplus \mathfrak{E}_0' \in \mathfrak{q}$ but $\bar{\in} \mathfrak{p}$, contradicting the maximality of the system $\mathfrak{E}_\lambda$ $(\lambda \in \Lambda)$. Thus $\mathfrak{E} \in \mathfrak{q}$ implies $\mathfrak{E} \dotplus \mathfrak{E}_0' \in \mathfrak{p}$. Conversely, since $\mathfrak{p} \subset \mathfrak{q}$ by assumption, $\mathfrak{E} \dotplus \mathfrak{E}_0' \in \mathfrak{p}$ implies

$$\mathfrak{E} \supset \mathfrak{E}_0(\mathfrak{E} \dotplus \mathfrak{E}_0') \in \mathfrak{q}$$

and hence $\mathfrak{E} \in \mathfrak{q}$ by 1). Therefore we obtain $\mathfrak{q} = \mathfrak{p}^{\mathfrak{E}_0}$ by the definition (3).

Theorem 45.2. If $\mathfrak{q} \supset \mathfrak{p} \in \mathcal{P}_{\mathfrak{M}}$, then we can find $\mathfrak{q}_0 \in \mathcal{P}_{\mathfrak{M}}$ such that $\mathfrak{q} \perp \mathfrak{q}_0$ and $\mathfrak{q}\mathfrak{q}_0 = \mathfrak{p}$.

Proof. By virtue of Theorem 45.1, if $\mathfrak{q} \supset \mathfrak{p} \in \mathcal{P}_{\mathfrak{M}}$, then we can find $\mathfrak{E}_0 \in \mathfrak{M}$ for which $\mathfrak{q} = \mathfrak{p}^{\mathfrak{E}_0}$. For such $\mathfrak{E}_0$ we have by the formula (7)

$$\mathfrak{p}^{\mathfrak{E}_0} \perp \mathfrak{p}^{\mathfrak{E}_0'}$$

and further by the formulas (8) and (5)

$$\mathfrak{p}^{\mathfrak{E}_0}\mathfrak{p}^{\mathfrak{E}_0'} = \mathfrak{p}^{\mathfrak{E}_0 \dotplus \mathfrak{E}_0'} = \mathfrak{p}^{\mathfrak{U}} = \mathfrak{p}.$$

Theorem 45.3. For every sequence $\mathfrak{p}_\nu \in \mathcal{P}_{\mathfrak{M}}$ $(\nu = 1, 2, \ldots)$ we also have $\prod_{\nu=1}^{\infty} \mathfrak{p}_\nu \in \mathcal{P}_{\mathfrak{M}}$.

Proof. For a system $\mathfrak{E}_\lambda \in \mathfrak{M}$ $(\lambda \in \Lambda)$, if $\mathfrak{E}_\lambda \mathfrak{E}_\rho = 0$ for

$\lambda \neq \rho$, then we can find by definition $\lambda_{\nu,\mu} \in \Lambda$ $(\nu, \mu = 1, 2, \ldots)$, such that $\mathfrak{E}_\lambda' \in \mathfrak{P}_\nu$ for $\lambda \neq \lambda_{\nu,\mu}$ $(\mu = 1, 2, \ldots)$, and hence

$$\mathfrak{E}_\lambda' \in \prod_{\nu=1}^{\infty} \mathfrak{P}_\nu \qquad \text{for } \lambda \neq \lambda_{\nu,\mu} \ (\nu, \mu = 1, 2, \ldots).$$

Therefore we have $\prod_{\nu=1}^{\infty} \mathfrak{P}_\nu \in \mathcal{P}_{\mathfrak{M}}$ by definition.

Let $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ be a spectrality on $\mathfrak{M}$ and $\mathfrak{R}$ the spectral ring of E, that is, the closed ring generated by $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$. Putting

$$E(\mathfrak{P}) = \bigwedge_{\mathfrak{E} \in \mathfrak{P}} E(\mathfrak{E}) \qquad \text{for } \mathfrak{P} \in \mathcal{P}_{\mathfrak{M}}, \tag{11}$$

we obtain a projection operator $E(\mathfrak{P}) \in \mathfrak{R}$ for every $\mathfrak{P} \in \mathcal{P}_{\mathfrak{M}}$. This system of projection operators $E(\mathfrak{P})$ $(\mathfrak{P} \in \mathcal{P}_{\mathfrak{M}})$ is called the derivative of a spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$.

Concerning the derivative, we have

$$E(\mathfrak{P}^{\mathfrak{E}}) = E(\mathfrak{E})E(\mathfrak{P}) \qquad \text{for } \mathfrak{P} \in \mathcal{P}_{\mathfrak{M}}, \ \mathfrak{E} \in \mathfrak{M}. \tag{12}$$

In fact, for each $\mathfrak{E}_0 \in \mathfrak{M}$ we have by the definitions (3) and (11)

$$E(\mathfrak{P}^{\mathfrak{E}_0}) = \bigwedge_{\mathfrak{E} \dotplus \mathfrak{E}_0' \in \mathfrak{P}} E(\mathfrak{E}) \geqq \bigwedge_{\mathfrak{E} \dotplus \mathfrak{E}_0' \in \mathfrak{P}} E((\mathfrak{E} \dotplus \mathfrak{E}_0')\mathfrak{E}_0)$$
$$\geqq E(\mathfrak{E}_0) \bigwedge_{\mathfrak{E} \in \mathfrak{P}} E(\mathfrak{E}) = E(\mathfrak{E}_0)E(\mathfrak{P}).$$

On the other hand, since $\mathfrak{E}\,\mathfrak{E}_0 \dotplus \mathfrak{E}_0' \supset \mathfrak{E}$, we have $\mathfrak{E}\,\mathfrak{E}_0 \dotplus \mathfrak{E}_0' \in \mathfrak{P}$ for every $\mathfrak{E} \in \mathfrak{P}$, and hence

$$\bigwedge_{\mathfrak{E} \dotplus \mathfrak{E}_0' \in \mathfrak{P}} E(\mathfrak{E}) \leqq \bigwedge_{\mathfrak{E} \in \mathfrak{P}} E(\mathfrak{E}\,\mathfrak{E}_0) = E(\mathfrak{E}_0)E(\mathfrak{P}).$$

If $\mathfrak{P} \perp \mathfrak{Q}$, then we can find by definition $\mathfrak{E} \in \mathfrak{P}$ and $\mathfrak{F} \in \mathfrak{Q}$ such that $\mathfrak{E}\,\mathfrak{F} = 0$, and hence by the definition (11)

$$E(\mathfrak{P})E(\mathfrak{Q}) \leqq E(\mathfrak{E})E(\mathfrak{F}) = E(\mathfrak{E}\,\mathfrak{F}) = 0.$$

Thus we have

$$E(\mathfrak{P})E(\mathfrak{Q}) = 0 \qquad \text{for } \mathfrak{P} \perp \mathfrak{Q}. \tag{13}$$

For a sequence $\mathfrak{P}_\nu \in \mathcal{P}_{\mathfrak{M}}$, putting $\mathfrak{P}_0 = \prod_{\nu=1}^{\infty} \mathfrak{P}_\nu$, we also have $\mathfrak{P}_0 \in \mathcal{P}_{\mathfrak{M}}$ by Theorem 45.3, and hence we can find by Theorem 45.1 a sequence $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ such that

$$\mathfrak{P}_\nu = \mathfrak{P}_0^{\mathfrak{E}_\nu} \qquad (\nu = 1, 2, \ldots).$$

Then we have by (12)

$$\bigcup_{\nu=1}^{\infty} E(\mathfrak{P}_\nu) = \bigcup_{\nu=1}^{\infty} E(\mathfrak{E}_\nu)E(\mathfrak{P}_0) = E\Big(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu\Big)E(\mathfrak{P}_0)$$
$$= E\Big(\mathfrak{P}_0^{\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu}\Big).$$

Since $\prod_{\nu=1}^{\infty} \mathfrak{P}_\nu = \mathfrak{P}_0 \sum_{\nu=1}^{\infty} \mathfrak{F}_\nu$ by (8), we obtain hence

$$E\left(\prod_{\nu=1}^{\infty} \mathfrak{P}_\nu\right) = \bigcup_{\nu=1}^{\infty} E(\mathfrak{P}_\nu). \tag{14}$$

From this relation (14) we conclude immediately

$$E(\mathfrak{P}) \leqq E(\mathfrak{Q}) \qquad \text{for } \mathfrak{P} \supset \mathfrak{Q} \tag{15}$$

Theorem 45.4. For each $\chi \in R$, putting

$$\mathfrak{P} = \{ \mathfrak{F} : \quad C_{[\chi]} \leqq E(\mathfrak{F}) \}$$

for the spectral ring $\mathfrak{R}$ of E, we obtain $\mathfrak{P} \in \mathfrak{P}_m$ and

$$E(\mathfrak{P}) = C_{[\chi]}.$$

Proof. Putting $\mathfrak{P} = \{ \mathfrak{F} : \quad C_{[\chi]} \leqq E(\mathfrak{F}) \}$, we see easily that $\mathfrak{P}$ is a σ-ideal. For every system $\mathfrak{F}_\lambda \in \mathfrak{M}$ $(\lambda \in \Lambda)$ subject to the condition: $\mathfrak{F}_\lambda \mathfrak{F}_\rho = 0$ for $\lambda \neq \rho$, we can find by Theorem 14.7 a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \ldots)$ such that

$$C_{[\chi]} \bigcup_{\lambda \in \Lambda} E(\mathfrak{F}_\lambda) = C_{[\chi]} \bigcup_{\nu=1}^{\infty} E(\mathfrak{F}_{\lambda_\nu}).$$

If $\rho \neq \lambda_\nu$ for every $\nu = 1, 2, \ldots$, then we have

$$C_{[\chi]} E(\mathfrak{F}_\rho) = C_{[\chi]} \bigcup_{\lambda \in \Lambda} E(\mathfrak{F}_\lambda) E(\mathfrak{F}_\rho) = C_{[\chi]} \bigcup_{\nu=1}^{\infty} E(\mathfrak{F}_{\lambda_\nu}) E(\mathfrak{F}_\rho)$$
$$= 0,$$

and hence

$$E(\mathfrak{F}_\rho') = 1 - E(\mathfrak{F}_\rho) \geqq C_{[\chi]},$$

that is, $\mathfrak{F}_\rho' \in \mathfrak{P}$ for $\rho \neq \lambda_\nu$ $(\nu = 1, 2, \ldots)$. Thus $\mathfrak{P}$ is separable, namely $\mathfrak{P} \in \mathfrak{P}_m$.

For an arbitrary $\mathfrak{y} \in R$, denoting by $\mathfrak{K}$ the totality of those projection operators $P \in \mathfrak{R}$, for which we can find $\mathfrak{F} \in \mathfrak{M}$ such that

$$[\mathfrak{y}] P = [\mathfrak{y}] E(\mathfrak{F}),$$

we see by Theorem 14.3 that $\mathfrak{K}$ is a closed ring, and it is obvious that $E(\mathfrak{F}) \in \mathfrak{K}$ for every $\mathfrak{F} \in \mathfrak{M}$. Thus we conclude $\mathfrak{K} = \mathfrak{R}$, and hence we can find $\mathfrak{F} \in \mathfrak{M}$ for which

$$[\mathfrak{y}] E(\mathfrak{F}) = [\mathfrak{y}] C_{[\chi]}.$$

If $[\mathfrak{y}] \geqq [\chi]$, then for such $\mathfrak{F} \in \mathfrak{M}$ we have by the formulas (16') and (12) in §14

$$E(\mathfrak{F}) \geqq C_{[\mathfrak{y}]} E(\mathfrak{F}) = C_{[\mathfrak{y}] E(\mathfrak{F})} = C_{[\mathfrak{y}]} C_{[\chi]} = C_{[\chi]}.$$

Since $E(\mathfrak{P}) \geqq C_{[\chi]}$ by the definition (11), we conclude therefore

$$[y]\,E(\mathfrak{F}) = [y]\,C_{[x]} \qquad \text{for } [y] \geqq [x].$$

Putting $P_0 = E(\mathfrak{F}) - C_{[x]}$, we have by §14(16') for every $y \in R$

$$C_{[P_0 y]}\,C_{[x]} = P_0\,C_{[y]}\,C_{[x]} = 0,$$

and hence by Theorem 19.5

$$[x + P_0 y] = [x] + [P_0 y] \geqq [x] \text{ for every } y \in R.$$

Thus we obtain for every $y \in R$

$$([x] + [P_0 y])\,E(\mathfrak{F}) = ([x] + [P_0 y])\,C_{[x]},$$

as proved just above. Since

$$[x]\,E(\mathfrak{F}) = [x]\,C_{[x]}E(\mathfrak{F}) = [x]\,C_{[x]},$$

we have hence

$$[P_0 y](E(\mathfrak{F}) - C_{[x]}) = 0,$$

that is, $[P_0 y]P_0 = 0$ for every $y \in R$. From this relation we conclude by the formula §14(3) that $P_0 y = 0$ for every $y \in R$. Thus we have $E(\mathfrak{F}) = C_{[x]}$.

For every $\mathfrak{E} \in \mathfrak{M}$ we have by Theorem 14.1

$$E(\mathfrak{E}) = \bigcup_{x \in R} C_{[E(\mathfrak{E})x]}$$

and for any $x \in R$ we can find by Theorem 45.4 $\mathfrak{F} \in \mathfrak{P}_{\mathfrak{M}}$ such that

$$E(\mathfrak{F}) = C_{[E(\mathfrak{E})x]}, \qquad \mathfrak{F} \ni \mathfrak{E}.$$

Thus we have

$$(16) \qquad E(\mathfrak{E}) = \bigcup_{\mathfrak{F} \ni \mathfrak{E}} E(\mathfrak{F} \qquad \text{for every } \mathfrak{E} \in \mathfrak{M}.$$

Theorem 45.5. For every $\mathfrak{F} \in \mathfrak{P}_{\mathfrak{M}}$ we can find $a \in R$ such that

$$E(\mathfrak{F}) = C_{[a]}$$

for the spectral ring $\mathfrak{R}$ of $E(\mathfrak{F})$ $(\mathfrak{E} \in \mathfrak{M})$.

Proof. For any system $P_\lambda \in \mathfrak{R}$ $(\lambda \in \Lambda)$ we can find by Maximal Theorem a maximal system of elements $0 \neq x_\gamma \in R$ $(\gamma \in \Gamma)$ subject to the condition:

$$C_{[x_\gamma]}\,C_{[x_\rho]} = 0 \qquad \text{for } \gamma \neq \rho,$$
$$C_{[x_\gamma]} \leqq E(\mathfrak{F})P_{\lambda_\gamma} \qquad \text{for some } \lambda_\gamma \in \Lambda.$$

Then we have

$$E(\mathfrak{F}) \bigcup_{\lambda \in \Lambda} P_\lambda = \bigcup_{\gamma \in \Gamma} C_{[x_\gamma]}.$$

Because it is obvious that $E(\mathfrak{F}) \bigcup_{\lambda \in \Lambda} P_\lambda \geqq \bigcup_{\gamma \in \Gamma} C_{[x_\gamma]}$, and if

$$E(\mathfrak{F}) \bigcup_{\lambda \in \Lambda} P_\lambda \neq \bigcup_{\gamma \in \Gamma} C_{[x_\gamma]},$$

then we can find $\lambda \in \Lambda$ for which

$$E(\mathfrak{P})P_\lambda\left(1 - \bigcup_{\gamma\in\Gamma} C_{[x_\gamma]}\right) \neq 0,$$

and hence there exists by Theorem 14.1 $0 \neq x \in R$ for which

$$E(\mathfrak{P})P_\lambda\left(1 - \bigcup_{\gamma\in\Gamma} C_{[x_\gamma]}\right) \geqq C_{[x]},$$

contradicting the maximality of the system $x_\lambda \in R$ $(\lambda \in \Lambda)$.

For each $\gamma \in \Gamma$, putting

$$\mathfrak{P}_\gamma = \{\mathfrak{E} : C_{[x_\gamma]} \leqq E(\mathfrak{E})\},$$

we obtain by Theorem 45.4 $\mathfrak{P}_\gamma \in \mathcal{P}_m$ and

$$E(\mathfrak{P}_\gamma) = C_{[x_\gamma]}.$$

Since $\mathfrak{P}_\gamma \supset \mathfrak{P}$, we can find by Theorem 45.1 $\mathfrak{E}_\gamma \in m$ $(\gamma \in \Gamma)$ for which $\mathfrak{P}_\gamma = \mathfrak{P}^{\mathfrak{E}_\gamma}$, and hence we have by (12)

$$E(\mathfrak{P})E(\mathfrak{E}_\gamma) = C_{[x_\gamma]}.$$

By virtue of Maximal Theorem, we can find then a maximal system of point sets $\mathfrak{F}_\delta \in m$ $(\delta \in \Delta)$ such that

$$\mathfrak{F}_\delta \mathfrak{F}_\rho = 0 \qquad \text{for } \delta \neq \rho,$$

$$E(\mathfrak{P})E(\mathfrak{F}_\delta) = E(\mathfrak{P})E(\mathfrak{E}_{\gamma_\delta}) \quad \text{for some } \gamma_\delta \in \Gamma.$$

Since $\mathfrak{P}$ is separable by assumption, if such Δ is infinite and not countable, then we can find $\delta \in \Delta$ for which $\mathfrak{F}_\delta' \in \mathfrak{P}$. For such δ we have by (11) $E(\mathfrak{P}) \leqq E(\mathfrak{F}_\delta')$ and hence

$$E(\mathfrak{P})E(\mathfrak{F}_\delta) = E(\mathfrak{P})(1 - E(\mathfrak{F}_\delta')) = 0,$$

contradicting $E(\mathfrak{P})E(\mathfrak{E}_{\gamma_\delta}) = C_{[x_{\gamma_\delta}]} \neq 0$. Consequently such Δ is at most countable. If there exists $\Gamma \ni \gamma \neq \gamma_\delta$ $(\delta \in \Delta)$, then we have for such $\gamma \in \Gamma$

$$E(\mathfrak{P})E\left(\sum_{\delta\in\Delta} \mathfrak{F}_\delta\right)E(\mathfrak{E}_\gamma) = \bigcup_{\delta\in\Delta} C_{[x_{\gamma_\delta}]}C_{[x_\gamma]} = 0,$$

and hence

$$E(\mathfrak{P})E\left(\mathfrak{E}_\gamma\left(\sum_{\delta\in\Delta} \mathfrak{F}_\delta\right)'\right) = E(\mathfrak{P})E(\mathfrak{E}_\gamma),$$

contradicting the maximality of the system $\mathfrak{F}_\delta$ $(\delta \in \Delta)$. Therefore Γ also is at most countable. Furthermore we have obviously

$$E(\mathfrak{P})\bigcup_{\lambda\in\Lambda} P_\lambda = E(\mathfrak{P})\bigcup_{\gamma\in\Gamma} P_{\lambda_\gamma}.$$

Therefore we conclude by Theorem 14.7 that there exists $a \in R$ for which

$$E(\mathfrak{P}) = C_{[a]}.$$

For a projection operator $P_0 \in \mathcal{R}$, putting

$$\mathfrak{P} = \{\mathfrak{E} : P_0 \leqq E(\mathfrak{E})\},$$

we see easily that $\mathfrak{P}$ is a σ-ideal. If $\mathfrak{P}$ is separable, then we can find by Theorem 45.5 an element $a \in R$ for which $E(\mathfrak{P}) = C_{[a]}$. For such $a \in R$ we have obviously by the definition (11)

$$P_0 \leqq E(\mathfrak{P}) = C_{[a]}$$

and hence by §14(16')

$$P_0 = P_0 C_{[a]} = C_{[P_0 a]}.$$

Thus we obtain by Theorem 45.4

$$E(\mathfrak{P}) = C_{[P_0 a]} = P_0.$$

Therefore we can state

Theorem 45.6. *For a projection operator* $P \in \mathfrak{R}$, *if*

$$\mathfrak{P} = \{\mathfrak{E} : P \leqq E(\mathfrak{E})\}$$

is a separable σ*-ideal, then we have* $E(\mathfrak{P}) = P$.

Recalling Theorem 14.7, we conclude easily by Theorem 45.5

Theorem 45.7. *For the spectral ring* $\mathfrak{R}$ *of* $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ *we have*

$$\{E(\mathfrak{E})E(\mathfrak{P}) : \mathfrak{E} \in \mathfrak{M}\} = E(\mathfrak{P})\mathfrak{R} \quad \text{for } \mathfrak{P} \in \mathfrak{P}_{\mathfrak{M}}.$$

Theorem 45.8. *For a commutative system of projection operators* $F(\mathfrak{P})$ $(\mathfrak{P} \in \mathfrak{P}_{\mathfrak{M}})$, *if*

$$\bigcup_{\mathfrak{P} \in \mathfrak{P}_{\mathfrak{M}}} F(\mathfrak{P}) = 1,$$

$$F(\mathfrak{P})F(\mathfrak{Q}) = 0 \quad \text{for } \mathfrak{P} \perp \mathfrak{Q},$$

$$F\Big(\prod_{\nu=1}^{\infty} \mathfrak{P}_\nu\Big) = \bigcup_{\nu=1}^{\infty} F(\mathfrak{P}_\nu)$$

for every sequence $\mathfrak{P}_\nu \in \mathfrak{P}_{\mathfrak{M}}$ $(\nu = 1, 2, \ldots)$, *then there exists uniquely a spectrality* $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ *such that* $F(\mathfrak{P})$ $(\mathfrak{P} \in \mathfrak{P}_{\mathfrak{M}})$ *coincides with the derivative* $E(\mathfrak{P})$ $(\mathfrak{P} \in \mathfrak{P}_{\mathfrak{M}})$ *of* E.

Proof. Putting

$$E(\mathfrak{E}) = \bigcup_{\mathfrak{P} \supset \mathfrak{E}} F(\mathfrak{P}) \qquad \text{for every } \mathfrak{E} \in \mathfrak{M},$$

we obtain a spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$. In fact, since $\Omega \in \mathfrak{P}$ for every $\mathfrak{P} \in \mathfrak{P}_{\mathfrak{M}}$, we have by assumption

$$E(\Omega) = \bigcup_{\mathfrak{P} \supset \Omega} F(\mathfrak{P}) = 1.$$

For any sequence $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ we have by assumption

$$E\left(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu\right) = \bigcup_{\mathfrak{f} \ni \sum_{\nu=1}^{\infty} \mathfrak{E}_\nu} F(\mathfrak{f}) \geqq \bigcup_{\mathfrak{f}_\nu \ni \mathfrak{E}_\nu \, (\nu=1,2,\ldots)} F\left(\prod_{\nu=1}^{\infty} \mathfrak{f}_\nu\right)$$
$$= \bigcup_{\nu=1}^{\infty} \bigcup_{\mathfrak{f} \ni \mathfrak{E}_\nu} F(\mathfrak{f}) = \bigcup_{\nu=1}^{\infty} E(\mathfrak{E}_\nu)$$

because $\mathfrak{f}_\nu \ni \mathfrak{E}_\nu$ $(\nu = 1, 2, \ldots)$ implies $\mathfrak{f}_\mu \ni \sum_{\nu=1}^{\infty} \mathfrak{E}_\nu$ for every $\mu = 1, 2, \ldots$. On the other hand, for every $\mathfrak{f} \ni \sum_{\nu=1}^{\infty} \mathfrak{E}_\nu$ we have by the formulas (5), (8), and (6)

$$\mathfrak{r} = \mathfrak{f} \sum_{\nu=1}^{\infty} \mathfrak{E}_\nu = \prod_{\nu=1}^{\infty} \mathfrak{f}^{\mathfrak{E}_\nu},$$
$$\mathfrak{f}^{\mathfrak{E}_\nu} \ni \mathfrak{E}_\nu \qquad (\nu = 1, 2, \ldots),$$

and hence

$$\bigcup_{\mathfrak{f} \ni \sum_{\nu=1}^{\infty} \mathfrak{E}_\nu} F(\mathfrak{f}) \leqq \bigcup_{\mathfrak{f}_\nu \ni \mathfrak{E}_\nu \, (\nu=1,2,\ldots)} F\left(\prod_{\nu=1}^{\infty} \mathfrak{f}_\nu\right).$$

Thus we obtain for every sequence $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$

$$E\left(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu\right) = \bigcup_{\nu=1}^{\infty} E(\mathfrak{E}_\nu).$$

If $\mathfrak{E}\mathfrak{F} = 0$, then we have by definition that $\mathfrak{f} \ni \mathfrak{E}$, $\mathfrak{q} \ni \mathfrak{F}$ implies $\mathfrak{f} \perp \mathfrak{q}$, and hence $F(\mathfrak{f})F(\mathfrak{q}) = 0$ by assumption. Thus we have for $\mathfrak{E}\mathfrak{F} = 0$

$$E(\mathfrak{E})E(\mathfrak{F}) = \bigcup_{\mathfrak{f} \ni \mathfrak{E}, \, \mathfrak{q} \ni \mathfrak{F}} F(\mathfrak{f})F(\mathfrak{q}) = 0.$$

Since $E(\mathfrak{E}) \cup E(\mathfrak{E}') = E(\Omega) = 1$ as proved just above, we obtain hence

$$E(\mathfrak{E}) + E(\mathfrak{E}') = 1 \qquad \text{for every } \mathfrak{E} \in \mathfrak{M}.$$

Therefore $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ is a spectrality on $\mathfrak{M}$ by definition.

Since $F(\mathfrak{f}) \leqq E(\mathfrak{E})$ for $\mathfrak{f} \ni \mathfrak{E}$, we have by the definition (11)

$$F(\mathfrak{f}) \leqq E(\mathfrak{f}) \qquad \text{for every } \mathfrak{f} \in \mathfrak{P}_{\mathfrak{M}}.$$

Let $\mathfrak{f}_0 \in \mathfrak{P}_{\mathfrak{M}}$ be arbitrary. For every $\mathfrak{f} \in \mathfrak{P}_{\mathfrak{M}}$, putting

$$\mathfrak{q} = \mathfrak{f}_0 \mathfrak{f},$$

we can find by Theorem 45.1 $\mathfrak{E}_0$, $\mathfrak{E} \in \mathfrak{M}$ for which

$$\mathfrak{f}_0 = \mathfrak{q}^{\mathfrak{E}_0}, \qquad \mathfrak{f} = \mathfrak{q}^{\mathfrak{E}}.$$

Then we have by the formulas (9), (4), and (8)

$$\mathfrak{q}^{\mathfrak{E}_0 \mathfrak{E}} = \mathfrak{f}_0^{\mathfrak{E}} \supset \mathfrak{f}_0,$$
$$\mathfrak{q}^{\mathfrak{E}_0 \mathfrak{E}} \mathfrak{q}^{\mathfrak{E}_0' \mathfrak{E}} = \mathfrak{q}^{\mathfrak{E}} = \mathfrak{f},$$

and $\mathfrak{f}_0 = \mathfrak{q}^{\mathfrak{E}_0} \perp \mathfrak{q}^{\mathfrak{E}_0' \mathfrak{E}}$ by definition. Thus we have by assumption

$$F(\mathfrak{f}) = F(\mathfrak{q}^{\mathfrak{E}}) = F(\mathfrak{q}^{\mathfrak{E}_0 \mathfrak{E}}) \cup F(\mathfrak{q}^{\mathfrak{E}_0' \mathfrak{E}})$$
$$\leqq F(\mathfrak{f}_0 \mathfrak{q}^{\mathfrak{E}_0 \mathfrak{E}}) \cup E(\mathfrak{q}^{\mathfrak{E}_0' \mathfrak{E}}) \leqq F(\mathfrak{f}_0) \cup (1 - E(\mathfrak{f}_0)),$$

because $E(\mathfrak{z}_0)E(\eta^{\mathfrak{T}_0'\mathfrak{T}}) = E(\eta^{\mathfrak{T}_0})E(\eta^{\mathfrak{T}_0'\mathfrak{T}}) = 0$. Since $\mathfrak{z}$ may be arbitrary, we obtain hence by assumption

$$F(\mathfrak{z}_0) \cup (1 - E(\mathfrak{z}_0)) = 1.$$

On the other hand we have

$$F(\mathfrak{z}_0)(1 - E(\mathfrak{z}_0)) \leqq E(\mathfrak{z}_0)(1 - E(\mathfrak{z}_0)) = 0.$$

Therefore we conclude

$$F(\mathfrak{z}_0) + (1 - E(\mathfrak{z}_0)) = 1,$$

and hence $F(\mathfrak{z}_0) = E(\mathfrak{z}_0)$. The uniqueness of such spectrality $E(\mathfrak{T})$ $(\mathfrak{T} \in \mathfrak{M})$ is obvious by the formula (16).

§46 Isomorphisms of Spectralities

A spectrality $E(\mathfrak{T})$ $(\mathfrak{T} \in \mathfrak{M})$ is said to be *isomorphic* to a spectrality $F(\mathfrak{T})$ $(\mathfrak{T} \in \mathfrak{M})$, if $E(\mathfrak{T}) = 0$ is equivalent to the relation $F(\mathfrak{T}) = 0$. $E(\mathfrak{T})$ $(\mathfrak{T} \in \mathfrak{M})$ is said to be *completely isomorphic* to $F(\mathfrak{T})$ $(\mathfrak{T} \in \mathfrak{M})$, if

$$E(\mathfrak{z}) = 0 \text{ is equivalent to } F(\mathfrak{z}) = 0$$

for the derivaltives $E(\mathfrak{z})$ and $F(\mathfrak{z})$ $(\mathfrak{z} \in \mathfrak{P}_{\mathfrak{M}})$. With this definition it is obvious by the formula §45(16) that if $E(\mathfrak{T})$ $(\mathfrak{T} \in \mathfrak{M})$ is completely isomorphic to $F(\mathfrak{T})$ $(\mathfrak{T} \in \mathfrak{M})$, then E is isomorphic to F.

Theorem 46.1. *For the spectral ring* $\mathfrak{R}$ *of* $E(\mathfrak{T})$ $(\mathfrak{T} \in \mathfrak{M})$ *and that* $\mathfrak{r}$ *of* $F(\mathfrak{T})$ $(\mathfrak{T} \in \mathfrak{M})$, *there exists an isomorphism* $P \to P^{\mathfrak{r}} \in \mathfrak{r}$ $(P \in \mathfrak{R})$ *such that*

$$F(\mathfrak{T}) = E(\mathfrak{T})^{\mathfrak{r}} \qquad \text{for every } \mathfrak{T} \in \mathfrak{M},$$

if and only if E *is completely isomorphic to* F.

Proof. We suppose that E is completely isomorphic to F. Then we have $E(\mathfrak{z})E(\eta) = 0$ if and only if $F(\mathfrak{z})F(\eta) = 0$. In fact, putting $\mathfrak{z}_0 = \mathfrak{z}\eta$, we can find by Theorem 45.1 $\mathfrak{T}$, $\mathfrak{T} \in \mathfrak{M}$ such that

$$\mathfrak{z} = \mathfrak{z}_0^{\mathfrak{T}}, \qquad \eta = \mathfrak{z}_0^{\mathfrak{T}},$$

and we have by the formula §45(12)

$$E(\mathfrak{z})E(\eta) = E(\mathfrak{T})E(\mathfrak{T})E(\mathfrak{z}_0) = E(\mathfrak{z}_0^{\mathfrak{T}\mathfrak{T}}),$$
$$F(\mathfrak{z})F(\eta) = F(\mathfrak{T})F(\mathfrak{T})F(\mathfrak{z}_0) = F(\mathfrak{z}_0^{\mathfrak{T}\mathfrak{T}}).$$

Furthermore we have by assumption $E(\mathfrak{z}_0^{\mathfrak{E}\mathfrak{F}}) = 0$ if and only if $F(\mathfrak{z}_0^{\mathfrak{E}\mathfrak{F}}) = 0$.

By virtue of Maximal Theorem, we can find a maximal system of separable σ-ideals $\mathfrak{z}_\lambda \in \mathfrak{P}_{\mathfrak{M}}$ $(\lambda \in \Lambda)$ subject to the condition:

$$E(\mathfrak{z}_\lambda) \neq 0 \qquad \text{for every } \lambda \in \Lambda,$$

$$E(\mathfrak{z}_\lambda)E(\mathfrak{z}_\rho) = 0 \qquad \text{for } \lambda \neq \rho.$$

Then we have

$$\bigcup_{\lambda\in\Lambda} E(\mathfrak{z}_\lambda) = 1.$$

Because, if $\bigcup_{\lambda\in\Lambda} E(\mathfrak{z}_\lambda) \neq 1$, then there exists $0 \neq a \in R$ for which

$$1 - \bigcup_{\lambda\in\Lambda} E(\mathfrak{z}_\lambda) \geqq C^{\mathfrak{R}}_{[a]_{\mathfrak{R}}},$$

and for such $a \in R$ we can find by Theorem 45.4 $\mathfrak{z} \in \mathfrak{P}_{\mathfrak{M}}$ for which

$$E(\mathfrak{z}) = C^{\mathfrak{R}}_{[a]_{\mathfrak{R}}},$$

contradicting the maximality of the system $\mathfrak{z}_\lambda \in \mathfrak{P}_{\mathfrak{M}}$ $(\lambda \in \Lambda)$. We also can prove likewise

$$\bigcup_{\lambda\in\Lambda} F(\mathfrak{z}_\lambda) = 1,$$

because E is completely isomorphic to F by assumption.

For every $\lambda \in \Lambda$, we have obviously

$$E(\mathfrak{E})E(\mathfrak{z}_\lambda) = E(\mathfrak{F})E(\mathfrak{z}_\lambda)$$

if and only if

$$E(\mathfrak{E}\mathfrak{F}')E(\mathfrak{z}_\lambda) = E(\mathfrak{E}'\mathfrak{F})E(\mathfrak{z}_\lambda) = 0,$$

and hence by §45(12) if and only if

$$E(\mathfrak{z}_\lambda^{\mathfrak{E}\mathfrak{F}'}) = E(\mathfrak{z}_\lambda^{\mathfrak{E}'\mathfrak{F}}) = 0.$$

Thus we conclude by assumption that we have

$$E(\mathfrak{E})E(\mathfrak{z}_\lambda) = E(\mathfrak{F})E(\mathfrak{z}_\lambda)$$

if and only if

$$F(\mathfrak{E})F(\mathfrak{z}_\lambda) = F(\mathfrak{F})F(\mathfrak{z}_\lambda).$$

Therefore, putting

$$F(\mathfrak{E})F(\mathfrak{z}_\lambda) = (E(\mathfrak{E})E(\mathfrak{z}_\lambda))^{\mathfrak{V}} \qquad (\mathfrak{E} \in \mathfrak{M}),$$

we obtain by Theorem 45.7 an isomorphism

$$P^{\mathfrak{V}} \in F(\mathfrak{z}_\lambda)\mathfrak{V} \qquad (P \in E(\mathfrak{z}_\lambda)\mathfrak{R}).$$

Furthermore, putting

$$P^{\mathfrak{V}} = \bigcup_{\lambda\in\Lambda} (E(\mathfrak{z}_\lambda)P)^{\mathfrak{V}} \qquad (P \in \mathfrak{R}),$$

we see easily that $P \overset{\mathfrak{V}}{\sim} P^{\mathfrak{V}}$ $(P \in \mathfrak{R})$ is an isomorphism such that

$$F(\mathfrak{E})F(\mathfrak{z}_\lambda) = (E(\mathfrak{E})E(\mathfrak{z}_\lambda))^{\mathfrak{V}} \qquad \text{for every } \mathfrak{E} \in \mathfrak{M},\ \lambda \in \Lambda.$$

Since $\bigcup_{\lambda\in\Lambda} E(\mathfrak{F}_\lambda) = \bigcup_{\lambda\in\Lambda} F(\mathfrak{F}_\lambda) = 1$, we obtain hence

$$F(\mathfrak{X}) = E(\mathfrak{X})^{\mathfrak{T}} \qquad \text{for every } \mathfrak{X}\in\mathfrak{M}.$$

Conversely if there exists such an isomorphism $P^{\mathfrak{T}}\in\mathfrak{T}$ $(P\in\mathfrak{R})$, then we see easily by the definition §45(11) that E is completely isomorphic to F.

Theorem 46.2. *For a spectrality $E(\mathfrak{X})$ $(\mathfrak{X}\in\mathfrak{M})$, if there is a separable σ-ideal $\mathfrak{F}_0\in\mathcal{P}_{\mathfrak{M}}$ for which $E(\mathfrak{F}_0)=1$, then every spectrality $F(\mathfrak{X})$ $(\mathfrak{X}\in\mathfrak{M})$ which is isomorphic to E, is completely isomorphic to E.*

Proof. If F is isomorphic to E, then we also have

$$F(\mathfrak{F}_0) = 1.$$

Because, if $F(\mathfrak{F}_0)\neq 1$, then we can find by the definition §45(11) a set $\mathfrak{X}\in\mathfrak{F}_0$ for which $F(\mathfrak{X})\neq 1$, and hence

$$F(\mathfrak{X}') = 1 - F(\mathfrak{X}) \neq 0.$$

Then we also have by assumption $E(\mathfrak{X}')\neq 0$, contradicting

$$E(\mathfrak{X}') = 1 - E(\mathfrak{X}) \leqq 1 - E(\mathfrak{F}_0) = 0.$$

For each $\mathfrak{F}\in\mathcal{P}_{\mathfrak{M}}$, putting $\eta = \mathfrak{F}_0\mathfrak{F}$ we can find by Theorem 45.1 $\mathfrak{X}_0$, $\mathfrak{X}\in\mathfrak{M}$ for which

$$\mathfrak{F}_0 = \eta^{\mathfrak{X}_0}, \qquad \mathfrak{F} = \eta^{\mathfrak{X}}.$$

Then we have by §45(2)

$$\begin{aligned} E(\mathfrak{F}) &= E(\mathfrak{X})E(\eta) = E(\mathfrak{X}\mathfrak{X}_0)E(\eta) + E(\mathfrak{X}\mathfrak{X}_0')E(\eta) \\ &= E(\mathfrak{X})E(\eta^{\mathfrak{X}_0}) + E(\mathfrak{X})\left(E(\eta) - E(\eta^{\mathfrak{X}_0})\right) = E(\mathfrak{X}), \end{aligned}$$

because $E(\eta^{\mathfrak{X}_0}) = E(\mathfrak{F}_0) = 1$. We also can prove likewise

$$F(\mathfrak{F}) = F(\mathfrak{X}).$$

Thus we have $E(\mathfrak{F}) = 0$ if and only if $F(\mathfrak{F}) = 0$.

Let $E(\mathfrak{X})$ $(\mathfrak{X}\in\mathfrak{M})$ be a spectrality and $\mathfrak{R}$ its spectral ring. For the minimal multiplicity $m_{\mathfrak{R}}(P)$ $(P\in\mathfrak{R})$ of $\mathfrak{R}$, putting

$$(1) \qquad m(\mathfrak{F}) = m_{\mathfrak{R}}(E(\mathfrak{F})) \qquad (\mathfrak{F}\in\mathcal{P}_{\mathfrak{M}}),$$

we obtain a cardinal valued function m on the property $\mathcal{P}_{\mathfrak{M}}$ of $\mathfrak{M}$. This function $m(\mathfrak{F})$ $(\mathfrak{F}\in\mathcal{P}_{\mathfrak{M}})$ is called the multiplicity function of the spectrality $E(\mathfrak{X})$ $(\mathfrak{X}\in\mathfrak{M})$.

For each $P\in\mathfrak{R}$ we have by Theorem 14.1 and the formula §14(15)

$$P = \bigcup_{x\in R} C_{[Px]}$$

and for every $\chi \in R$ we can find by Theorem 45.4 $\mathfrak{x} \in \mathfrak{P}_{\mathfrak{M}}$ for which

$$C_{[P\chi]} = E(\mathfrak{x}).$$

Thus we obtain by the formula §44(4)

$$(2) \qquad m_{\mathfrak{R}}(P) = \operatorname*{Min}_{0 \neq E(\mathfrak{x}) \leq P} m(\mathfrak{x}) \qquad \text{for every } 0 \neq P \in \mathfrak{R}.$$

Theorem 46.3. For two spectralities $E(\mathfrak{X})$ and $F(\mathfrak{X})$ ($\mathfrak{X} \in \mathfrak{M}$), there exists a unitary operator U such that

$$F(\mathfrak{X}) = U E(\mathfrak{X}) U^* \qquad (\mathfrak{X} \in \mathfrak{M}),$$

if and only if we have

$$m_F(\mathfrak{x}) = m_E(\mathfrak{x}) \qquad (\mathfrak{x} \in \mathfrak{P}_{\mathfrak{M}}),$$

denoting by m_E the multiplicity function of E and by m_F that of F.

Proof. If there is a unitary operator U such that

$$F(\mathfrak{X}) = U E(\mathfrak{X}) U^* \qquad (\mathfrak{X} \in \mathfrak{M}),$$

then we see by the formulas §45(11) and §41(3) that we have

$$F(\mathfrak{x}) = U E(\mathfrak{x}) U^* \quad \text{for every } \mathfrak{x} \in \mathfrak{P}_{\mathfrak{M}}.$$

From this relation we conclude that $E(\mathfrak{x}) = 0$ is equivalent to the relation $F(\mathfrak{x}) = 0$. Thus, denoting by $\mathfrak{R}$ the spectral ring of E and by $\mathfrak{r}$ that of F, we can find by Theorem 46.1 an isomorphism

$$P \to P^{\mathfrak{r}} \in \mathfrak{r} \qquad (P \in \mathfrak{R})$$

such that

$$E(\mathfrak{X})^{\mathfrak{r}} = U E(\mathfrak{X}) U^* \qquad (\mathfrak{X} \in \mathfrak{M}).$$

Denoting by $\mathfrak{K}$ the totality of those projection operators $P \in \mathfrak{R}$ for which

$$P^{\mathfrak{r}} = U P U^*,$$

we see easily by the formulas (2) and (3) in §41 that $\mathfrak{K}$ is a closed ring containing all $E(\mathfrak{X})$ ($\mathfrak{X} \in \mathfrak{M}$). Thus we have $\mathfrak{K} = \mathfrak{R}$, and hence

$$P^{\mathfrak{r}} = U P U^* \qquad \text{for every } P \in \mathfrak{R}.$$

Therefore we obtain by Theorem 44.2 for every $\mathfrak{x} \in \mathfrak{P}_{\mathfrak{M}}$

$$m_F(\mathfrak{x}) = m_{\mathfrak{r}}(F(\mathfrak{x})) = m_{\mathfrak{R}}(E(\mathfrak{x})) = m_E(\mathfrak{x})$$

for the minimal multiplicities $m_{\mathfrak{R}}(P)$ ($P \in \mathfrak{R}$) and $m_{\mathfrak{r}}(P)$ ($P \in \mathfrak{r}$).

Conversely if $m_F(\mathfrak{x}) = m_E(\mathfrak{x})$ for every $\mathfrak{x} \in \mathfrak{P}_{\mathfrak{M}}$, then

$$m_E(\mathfrak{x}) = 0 \text{ is equivalent to } m_F(\mathfrak{x}) = 0,$$

that is,

$$E(\mathfrak{F}) = 0 \text{ is equivalent to } F(\mathfrak{F}) = 0.$$

Therefore there exists by Theorem 46.1 an isomorphism $P^{\mathfrak{T}} \in \mathfrak{T}$ $(P \in \mathfrak{R})$ such that

$$F(\mathfrak{E}) = E(\mathfrak{E})^{\mathfrak{T}} \qquad (\mathfrak{E} \in \mathfrak{M}).$$

Furthermore we have by (2) for every $P \in \mathfrak{R}$, $P \neq 0$,

$$m_{\mathfrak{T}}(P^{\mathfrak{T}}) = \operatorname*{Min}_{0 \neq F(\mathfrak{F}) \leq P^{\mathfrak{T}}} m_F(\mathfrak{F})$$

$$= \operatorname*{Min}_{0 \neq E(\mathfrak{F}) \leq P} m_E(\mathfrak{F}) = m_{\mathfrak{R}}(P).$$

Therefore there exists by Theorem 44.2 a unitary operator U such that

$$P^{\mathfrak{T}} = U P U^* \qquad \text{for every } P \in \mathfrak{R},$$

and hence

$$F(\mathfrak{E}) = U E(\mathfrak{E}) U^* \qquad \text{for every } \mathfrak{E} \in \mathfrak{M}.$$

§47 Unitary Transformation of Dilatators

Let $\mathfrak{R}$ be a closed ring of projection operators containing 1 in a Hilbert space R and $\mathfrak{L}_{\mathfrak{R}}$ the proper space of $\mathfrak{R}$. For a unitary operator U on R, putting

$$\mathfrak{R}^U = \{ U P U^* : \quad P \in \mathfrak{R} \}$$

we obtain a closed ring $\mathfrak{R}^U$ containing 1 and

$$U P U^* \in \mathfrak{R}^U \qquad (P \in \mathfrak{R})$$

is an isomorphism, as stated in §41. Thus there exists by Theorem 41.1 a homeomorphic correspondence

$$\mathfrak{F}^U \in \mathfrak{L}_{\mathfrak{R}^U} \qquad (\mathfrak{F} \in \mathfrak{L}_{\mathfrak{R}})$$

for the proper space $\mathfrak{L}_{\mathfrak{R}^U}$ of $\mathfrak{R}^U$ such that

$$U P U^* \in \mathfrak{F}^U \quad \text{if and only if} \quad P \in \mathfrak{F}.$$

For a dilatator T with domain D by $\mathfrak{R}$ we see easily by definition that $U T U^*$ is a dilatator with domain UD by $\mathfrak{R}^U$, and we have by Theorem 24.1

$$(U T U^*, \mathfrak{F}^U) = (T, \mathfrak{F}) \qquad \text{for every } \mathfrak{F} \in \mathfrak{L}_{\mathfrak{R}},$$

because we have for every $0 \neq x \in D$ and $P \in \mathfrak{R}$

$$\frac{((U T U^*)(U P U^*) U x, U x)}{\| (U P U^*) U P x \|^2} = \frac{(T P x, x)}{\| P x \|^2}.$$

Thus we see by §23(1) for the spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L})$ of T that $U E(\mathfrak{E}) U^*$ $(\mathfrak{E} \in \mathfrak{L})$ is the spectrality of $U T U^*$.

Furthermore we see by §30(1) for the spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L}^{\Gamma})$ of a system of dilatators T_γ $(\gamma \in \Gamma)$ by $\mathfrak{R}$ that

$$U E(\mathfrak{E}) U^* \qquad (\mathfrak{E} \in \mathfrak{L}^{\Gamma})$$

is the spectrality of the system of dilatators $U T_\gamma U^*$ $(\gamma \in \Gamma)$ by the ring $\mathfrak{R}^U$.

Conversely, for the spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L}^{\Gamma})$ of a system of dilatators T_γ $(\gamma \in \Gamma)$ by a closed ring $\mathfrak{R}$ and for the spectrality $F(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L}^{\Gamma})$ of a system of dilatators S_γ $(\gamma \in \Gamma)$ by a closed $\mathfrak{R}$ there exists a unitary operator U such that

$$F(\mathfrak{E}) = U E(\mathfrak{E}) U^* \quad \text{for every } \mathfrak{E} \in \mathfrak{L}^{\Gamma},$$

then we have

$$S_{\gamma_0} = U T_{\gamma_0} U^* \qquad \text{for every } \gamma_0 \in \Gamma,$$

because we have by §30(6)

$$S_{\gamma_0} = \int \zeta_{\gamma_0} F(d(\zeta_\gamma)_{\gamma \in \Gamma}),$$

$$U T_{\gamma_0} U^* = \int \zeta_{\gamma_0} U E(d(\zeta_\gamma)_{\gamma \in \Gamma}) U^*,$$

since $U E(\mathfrak{E}) U^*$ $(\mathfrak{E} \in \mathfrak{L}^{\Gamma})$ is the spectrality of $U T_\gamma U^*$ $(\gamma \in \Gamma)$, as proved just above.

Therefore we can state

Theorem 47.1. Let $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L}^{\Gamma})$ be the spectrality of a system of dilatators T_γ $(\gamma \in \Gamma)$ by a closed ring $\mathfrak{R}$ and $F(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L}^{\Gamma})$ the spectrality of S_γ $(\gamma \in \Gamma)$ by $\mathfrak{R}$. For a unitary operator U we have

$$S_\gamma = U T_\gamma U^* \qquad \text{for every } \gamma \in \Gamma,$$

if and only if

$$F(\mathfrak{E}) = U E(\mathfrak{E}) U^* \quad \text{for every } \mathfrak{E} \in \mathfrak{L}^{\Gamma}.$$

Recalling the definition of functions of dilatators in §30, we conclude immediately from this Theorem 47.1

Theorem 47.2. Let T_γ $(\gamma \in \Gamma)$ be a system of dilatators by a closed ring $\mathfrak{R}$ and U a unitary operator. For any $(\mathfrak{L}^{\Gamma})$-measurable function $\varphi(\zeta)$ $(\zeta \in \mathfrak{C}^{\Gamma})$ we have

$$U \varphi(T_\gamma)_{\gamma \in \Gamma} U^* = \varphi(U T_\gamma U^*)_{\gamma \in \Gamma}.$$

Let $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L}^{\Gamma})$ be the spectrality of a system of dilatators T_γ $(\gamma \in \Gamma)$ by a closed ring $\mathfrak{R}$. The multiplicity function

$$m(\mathfrak{F}) \qquad (\mathfrak{F} \in \mathfrak{P}_{\mathfrak{L}^{\Gamma}})$$

of E on the property $\mathfrak{P}_{\mathfrak{L}^{\Gamma}}$ of the Borel class $\mathfrak{L}^{\Gamma}$ is called the <u>multiplicity function</u> of the system T_γ $(\gamma \in \Gamma)$. With this definition we conclude by Theorem 47.1 immediately from Theorem 46.3

<u>Theorem</u> 47.3. <u>For two systems of dilatators</u> T_γ $(\gamma \in \Gamma)$ <u>by a closed ring</u> $\mathfrak{R}$ <u>and</u> S_γ $(\gamma \in \Gamma)$ <u>by</u> $\mathfrak{R}$, <u>we can find a unitary operator</u> U <u>such that</u>

$$S_\gamma = U T_\gamma U^* \qquad \text{<u>for every</u> } \gamma \in \Gamma,$$

<u>if and only if the multiplicity function of</u> T_γ $(\gamma \in \Gamma)$ <u>coincides with that of</u> S_γ $(\gamma \in \Gamma)$.

For the spectrality $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{L})$ of a regular closed normal operator N, the multiplicity function $m(\mathfrak{F})$ $(\mathfrak{F} \in \mathfrak{P}_{\mathfrak{L}})$ of E is called the <u>multiplicity function</u> of N. With this definition we have obviously by Theorem 47.3

<u>Theorem</u> 47.4. <u>For two regular closed normal operators</u> N_1 <u>and</u> N_2 <u>we can find a unitary operator</u> U <u>for which</u> $N_2 = U N_1 U^*$, <u>if and only if the multiplicity function of</u> N_1 <u>coincides with that of</u> N_2.

§48 <u>(L_2)-spaces</u>

Let $\mathfrak{M}$ be a totally additive set class in an abstract space Ω. A function $\mu(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ on $\mathfrak{M}$ is said to be a <u>measure</u> on $\mathfrak{M}$, if μ is positive and totally additive, that is,

1) $\quad 0 \leqq \mu(\mathfrak{E}) < +\infty \qquad$ for every $\mathfrak{E} \in \mathfrak{M}$,

2) $\quad \mu\left(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu\right) = \sum_{\nu=1}^{\infty} \mu(\mathfrak{E}_\nu)$ if $\mathfrak{E}_\nu \mathfrak{E}_\rho = 0$ for $\nu \neq \rho$.

For a measure μ on $\mathfrak{M}$ we see easily by definition that

$$\mu(0) = 0,$$

$$\mu(\mathfrak{E}) \leqq \mu(\mathfrak{F}) \qquad \text{for } \mathfrak{E} \subset \mathfrak{F},$$

and we have

$$\lim_{\nu \to \infty} \mu(\mathfrak{E}_\nu) = \mu(\mathfrak{E})$$

if $\mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \cdots$, $\mathfrak{E} = \sum_{\nu=1}^{\infty} \mathfrak{E}_\nu$ or if $\mathfrak{E}_1 \supset \mathfrak{E}_2 \supset \cdots$, $\mathfrak{E} = \prod_{\nu=1}^{\infty} \mathfrak{E}_\nu$.

Let μ be a measure on $\mathfrak{M}$ in the sequel. If a $(\mathfrak{M})$-measurable function φ on Ω is bounded in a point set $\mathfrak{E}_0 \in \mathfrak{M}$, then we define the integral of φ by μ in $\mathfrak{E}_0$ likewise as in §28 to mean

$$\int_{\mathfrak{E}_0} \varphi(\zeta)\mu(d\zeta) = \lim_{\varepsilon \to 0} \sum_{\nu=1}^{\varkappa} \varphi(\zeta_\nu)\mu(\mathfrak{E}_\nu)$$

for $\mathfrak{E}_0 = \sum_{\nu=1}^{\varkappa} \mathfrak{E}_\nu$, $\mathfrak{E}_\nu \mathfrak{E}_\rho = 0$ for $\nu \neq \rho$, $\mathfrak{E}_\nu \in \mathfrak{M}$,

$$\sup_{\xi, \eta \in \mathfrak{E}_\nu} |\varphi(\xi) - \varphi(\eta)| \leqq \varepsilon \qquad (\nu = 1, 2, \ldots, \varkappa).$$

With this definition we have obviously

$$\int_{\mathfrak{E}_0} \{\alpha\varphi(\zeta) + \beta\psi(\zeta)\}\mu(d\zeta) = \alpha \int_{\mathfrak{E}_0} \varphi(\zeta)\mu(d\zeta) + \beta \int_{\mathfrak{E}_0} \psi(\zeta)\mu(d\zeta),$$

$$\int_{\mathfrak{E}} \varphi(\zeta)\mu(d\zeta) + \int_{\mathfrak{F}} \varphi(\zeta)\mu(d\zeta) = \int_{\mathfrak{E}+\mathfrak{F}} \varphi(\zeta)\mu(d\zeta) \qquad \text{for } \mathfrak{E}\mathfrak{F} = 0,$$

$$\left| \int_{\mathfrak{E}} \varphi(\zeta)\mu(d\zeta) \right| \leqq \int_{\mathfrak{E}} |\varphi(\zeta)|\mu(d\zeta).$$

Let a $(\mathfrak{M})$-measurable function φ on Ω be positive on a set $\mathfrak{E}_0 \in \mathfrak{M}$, that is, $\varphi(\zeta) \geqq 0$ for every $\zeta \in \mathfrak{E}_0$. Denoting by $\mathfrak{K}$ the totality of those sets $\mathfrak{E} \in \mathfrak{M}$ for which φ is bounded in $\mathfrak{E}$, we define the integral of φ by μ in $\mathfrak{E}_0$ to mean

$$\int_{\mathfrak{E}_0} \varphi(\zeta)\mu(d\zeta) = \sup_{\mathfrak{E} \in \mathfrak{K}} \int_{\mathfrak{E}\mathfrak{E}_0} \varphi(\zeta)\mu(d\zeta).$$

With this definition we have obviously

$$\int_{\mathfrak{E}_0} \varphi(\zeta)\mu(d\zeta) \leqq \int_{\mathfrak{E}_0} \psi(\zeta)\mu(d\zeta)$$

if $0 \leqq \varphi(\zeta) \leqq \psi(\zeta)$ for every $\zeta \in \mathfrak{E}_0$,

$$\int_{\mathfrak{E}_0} \{\varphi(\zeta) + \psi(\zeta)\}\mu(d\zeta) = \int_{\mathfrak{E}_0} \varphi(\zeta)\mu(d\zeta) + \int_{\mathfrak{E}_0} \psi(\zeta)\mu(d\zeta),$$

$$\int_{\mathfrak{E}} \varphi(\zeta)\mu(d\zeta) + \int_{\mathfrak{F}} \varphi(\zeta)\mu(d\zeta) = \int_{\mathfrak{E}+\mathfrak{F}} \varphi(\zeta)\mu(d\zeta) \qquad \text{for } \mathfrak{E}\mathfrak{F} = 0,$$

$$\int_{\mathfrak{E}} \alpha\varphi(\zeta)\mu(d\zeta) = \alpha \int_{\mathfrak{E}} \varphi(\zeta)\mu(d\zeta) \qquad \text{for } \alpha \geqq 0.$$

Lemma 1. *If* $0 \leqq \varphi_1(\zeta) \leqq \varphi_2(\zeta) \leqq \cdots$ *and*

$$\lim_{\nu \to \infty} \varphi_\nu(\zeta) = \varphi(\zeta) \quad \text{for every } \zeta \in \mathfrak{E}_0,$$

then we have

$$\lim_{\nu\to\infty}\int_{\mathfrak{E}_0}\varphi_\nu(\zeta)\mu(d\zeta)=\int_{\mathfrak{E}_0}\varphi(\zeta)\mu(d\zeta).$$

Proof. If φ is bounded in $\mathfrak{E}_0$ and $\varphi(\zeta)\leqq\gamma$ for every $\zeta\in\mathfrak{E}_0$, then for any $\varepsilon>0$, putting

$$A_\nu=\{\zeta:\varphi(\zeta)-\varphi_\nu(\zeta)>\varepsilon\}\quad(\nu=1,2,\ldots),$$

we see easily by assumption that

$$A_1\supset A_2\supset\cdots,\qquad\prod_{\nu=1}^{\infty}A_\nu=0,$$

and hence $\lim_{\nu\to\infty}\mu(A_\nu)=0$. On the other hand we have for every $\nu=1,2,\ldots$

$$\begin{aligned}&\int_{\mathfrak{E}_0}\varphi(\zeta)\mu(d\zeta)-\int_{\mathfrak{E}_0}\varphi_\nu(\zeta)\mu(d\zeta)\\&=\int_{A_\nu}(\varphi(\zeta)-\varphi_\nu(\zeta))\mu(d\zeta)+\int_{\mathfrak{E}_0-A_\nu}(\varphi(\zeta)-\varphi_\nu(\zeta))\mu(d\zeta)\\&\leqq\gamma\mu(A_\nu)+\varepsilon\mu(\mathfrak{E}_0).\end{aligned}$$

Thus we obtain

$$\lim_{\nu\to\infty}\{\int_{\mathfrak{E}_0}\varphi(\zeta)\mu(d\zeta)-\int_{\mathfrak{E}_0}\varphi_\nu(\zeta)\mu(d\zeta)\}\leqq\varepsilon\mu(\mathfrak{E}_0).$$

Since $\varepsilon>0$ may be arbitrary, we conclude hence

$$\lim_{\nu\to\infty}\int_{\mathfrak{E}_0}\varphi_\nu(\zeta)\mu(d\zeta)=\int_{\mathfrak{E}_0}\varphi(\zeta)\mu(d\zeta).$$

In the general case, it is evident that

$$\int_{\mathfrak{E}_\varepsilon}\varphi_\nu(\zeta)\mu(d\zeta)\leqq\int_{\mathfrak{E}_0}\varphi(\zeta)\mu(d\zeta)$$

for every $\nu=1,2,\ldots$. For a set $\mathfrak{E}\in\mathfrak{M}$, if φ is bounded in $\mathfrak{E}$, then we have

$$\begin{aligned}\int_{\mathfrak{E}\mathfrak{E}_0}\varphi(\zeta)\mu(d\zeta)&=\lim_{\nu\to\infty}\int_{\mathfrak{E}\mathfrak{E}_0}\varphi_\nu(\zeta)\mu(d\zeta)\\&\leqq\lim_{\nu\to\infty}\int_{\mathfrak{E}_0}\varphi_\nu(\zeta)\mu(d\zeta),\end{aligned}$$

as proved just above. Thus we obtain by definition

$$\int_{\mathfrak{E}_0}\varphi(\zeta)\mu(d\zeta)\leqq\lim_{\nu\to\infty}\int_{\mathfrak{E}_0}\varphi_\nu(\zeta)\mu(d\zeta).$$

From this Lemma 1 we conclude immediately

Lemma 2. If $\varphi_1(\zeta)\geqq\varphi_2(\zeta)\geqq\cdots$, $\lim_{\nu\to\infty}\varphi_\nu(\zeta)=0$ for every $\zeta\in\mathfrak{E}_0$, and $\int_{\mathfrak{E}_0}\varphi_1(\zeta)\mu(d\zeta)<+\infty$, then we have

$$\lim_{\nu\to\infty}\int_{\mathfrak{E}_0}\varphi_\nu(\zeta)\mu(d\zeta)=0.$$

For a characteristic function $\chi_{\mathfrak{F}}$ of $\mathfrak{F}\in\mathfrak{M}$, we have obviously

$$\int_{\mathfrak{E}_0}\chi_{\mathfrak{F}}(\zeta)\varphi(\zeta)\mu(d\zeta)=\int_{\mathfrak{F}\mathfrak{E}_0}\varphi(\zeta)\mu(d\zeta).$$

Thus, as a special case of Lemma 2, we have

Lemma 3. If $\int_{\mathfrak{E}_0}\varphi(\zeta)\mu(d\zeta)<+\infty$, then for every sequence $\mathfrak{E}_\nu\in\mathfrak{M}$ such that $\mathfrak{E}_1\supset\mathfrak{E}_2\supset\cdots$ and $\prod_{\nu=1}^{\infty}\mathfrak{E}_\nu=0$, we have

$$\lim_{\nu\to\infty}\int_{\mathfrak{E}_0\mathfrak{E}_\nu}\varphi(\zeta)\mu(d\zeta)=0.$$

Lemma 4. If $0\leqq\varphi_1(\zeta)\leqq\varphi_2(\zeta)\leqq\cdots$ for every $\zeta\in\mathfrak{E}_0$ and

$$\lim_{\nu\to\infty}\int_{\mathfrak{E}_0}\varphi_\nu(\zeta)\mu(d\zeta)<+\infty,$$

then there exists a set $\Psi\in\mathfrak{M}$ such that $\Psi\subset\mathfrak{E}_0$, $\mu(\Psi)=\mu(\mathfrak{E}_0)$,

$$\lim_{\nu\to\infty}\varphi_\nu(\zeta)<+\infty \quad \text{for every } \zeta\in\Psi.$$

Proof. Putting

$$\Psi_{\rho,\nu}=\{\zeta:\ \varphi_\nu(\zeta)\leqq\rho,\ \zeta\in\mathfrak{E}_0\}\quad(\rho,\nu=1,2,\ldots),$$

we have obviously $\Psi_{\rho,1}\supset\Psi_{\rho,2}\supset\cdots$ $(\rho=1,2,\ldots)$ and

$$\int_{\mathfrak{E}_0}\varphi_\nu(\zeta)\mu(d\zeta)\geqq\int_{\mathfrak{E}_0-\Psi_{\rho,\nu}}\rho\,\mu(d\zeta)=\rho\,(\mu(\mathfrak{E}_0)-\mu(\Psi_{\rho,\nu})).$$

Thus, putting

$$\Psi_\rho=\prod_{\nu=1}^{\infty}\Psi_{\rho,\nu}\qquad(\rho=1,2,\ldots)$$

$$\gamma=\lim_{\nu\to\infty}\int_{\mathfrak{E}_0}\varphi_\nu(\zeta)\mu(d\zeta),$$

we have

$$\Psi_1\subset\Psi_2\subset\cdots,$$

$$\lim_{\nu\to\infty}\varphi_\nu(\zeta)\leqq\rho\quad\text{for every }\zeta\in\Psi_\rho,$$

$$\mu(\Psi_\rho)\geqq\mu(\mathfrak{E}_0)-\frac{\gamma}{\rho}\qquad(\rho=1,2,\ldots).$$

Therefore, putting $\Psi=\sum_{\rho=1}^{\infty}\Psi_\rho$, we obtain

$$\lim_{\nu\to\infty}\varphi_\nu(\zeta)<+\infty\quad\text{for every }\zeta\in\Psi,$$

and $\mu(\Psi)\geqq\mu(\mathfrak{E}_0)$. Since $\Psi\subset\mathfrak{E}_0$, we conclude hence

$$\mu(\Psi)=\mu(\mathfrak{E}_0).$$

For a $(\mathfrak{M})$-measurable function φ on Ω, if

$$\int_{\mathfrak{E}_0} |\varphi(\zeta)|\mu(d\zeta) < +\infty,$$

then for every two sequences $\mathfrak{E}_\nu$, $\mathfrak{F}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ such that

$$\mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \ldots, \qquad \sum_{\nu=1}^{\infty} \mathfrak{E}_\nu = \mathfrak{E}_0,$$

$$\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \ldots, \qquad \sum_{\nu=1}^{\infty} \mathfrak{F}_\nu = \mathfrak{E}_0,$$

and φ is bounded in $\mathfrak{E}_\nu$ as well as in $\mathfrak{F}_\nu$ for every $\nu = 1, 2, \ldots$, we have for every ν, $\rho = 1, 2, \ldots$

$$\left| \int_{\mathfrak{E}_\nu} \varphi(\zeta)\mu(d\zeta) - \int_{\mathfrak{F}_\rho} \varphi(\zeta)\mu(d\zeta) \right|$$

$$= \left| \int_{\mathfrak{E}_\nu \mathfrak{F}_\rho'} \varphi(\zeta)\mu(d\zeta) - \int_{\mathfrak{F}_\rho \mathfrak{E}_\nu'} \varphi(\zeta)\mu(d\zeta) \right|$$

$$\leqq \int_{\mathfrak{E}_0 \mathfrak{F}_\rho'} |\varphi(\zeta)|\mu(d\zeta) + \int_{\mathfrak{E}_0 \mathfrak{E}_\nu'} |\varphi(\zeta)|\mu(d\zeta).$$

Since

$$\mathfrak{E}_0\mathfrak{E}_1' \supset \mathfrak{E}_0\mathfrak{E}_2' \supset \ldots, \qquad \prod_{\nu=1}^{\infty} \mathfrak{E}_0\mathfrak{E}_\nu' = 0,$$

$$\mathfrak{E}_0\mathfrak{F}_1' \supset \mathfrak{E}_0\mathfrak{F}_2' \supset \ldots, \qquad \prod_{\rho=1}^{\infty} \mathfrak{E}_0\mathfrak{F}_\rho' = 0,$$

we conclude hence by Lemma 3

$$\lim_{\nu \to \infty} \int_{\mathfrak{E}_\nu} \varphi(\zeta)\mu(d\zeta) = \lim_{\rho \to \infty} \int_{\mathfrak{F}_\rho} \varphi(\zeta)\mu(d\zeta).$$

Therefore, in this case, we shall say that φ is integrable by μ in $\mathfrak{E}_0$ and define by this same limit the integral

$$\int_{\mathfrak{E}_0} \varphi(\zeta)\mu(d\zeta).$$

With this definition we have obviously that both φ and ψ are integrable by μ in $\mathfrak{E}_0$, then $\alpha\varphi(\zeta) + \beta\psi(\zeta)$ also is integrable by μ in $\mathfrak{E}_0$ for every complex numbers α, β and

$$\int_{\mathfrak{E}_0} \{\alpha\varphi(\zeta) + \beta\psi(\zeta)\}\mu(d\zeta)$$

$$= \alpha \int_{\mathfrak{E}_0} \varphi(\zeta)\mu(d\zeta) + \beta \int_{\mathfrak{E}_0} \psi(\zeta)\mu(d\zeta).$$

$$\int_{\mathfrak{E}} \varphi(\zeta)\mu(d\zeta) + \int_{\mathfrak{F}} \varphi(\zeta)\mu(d\zeta) = \int_{\mathfrak{E}+\mathfrak{F}} \varphi(\zeta)\mu(d\zeta)$$

if $\mathfrak{E}\mathfrak{F} = 0$ and φ is integrable by μ in $\mathfrak{E} + \mathfrak{F}$.

$$\left| \int_{\mathfrak{E}} \varphi(\zeta)\mu(d\zeta) \right| \leqq \int_{\mathfrak{E}} |\varphi(\zeta)|\mu(d\zeta).$$

We shall denote by L_2 the totality of those $(\mathfrak{M})$-measurable func-

tions φ on Ω for which

$$\int_{\Omega} |\varphi(\zeta)|^2 \mu(d\zeta) < +\infty .$$

With this definition we see easily that $\varphi \in L_2$ implies $\alpha\varphi \in L_2$ for every complex number α, and that $\varphi, \psi \in L_2$ implies $\varphi + \psi \in L_2$, because

$$|\varphi(\zeta) + \psi(\zeta)|^2 \leqq 2\{|\varphi(\zeta)|^2 + |\psi(\zeta)|^2\}.$$

For every $\varphi, \psi \in L_2$, since

$$|\varphi(\zeta)\overline{\psi(\zeta)}| \leqq \frac{1}{2}\{|\varphi(\zeta)|^2 + |\psi(\zeta)|^2\},$$

we have

$$\int_{\Omega} |\varphi(\zeta)\overline{\psi(\zeta)}| \mu(d\zeta) < +\infty .$$

Thus we define (φ, ψ) for $\varphi, \psi \in L_2$ to mean

$$(\varphi, \psi) = \int_{\Omega} \varphi(\zeta)\overline{\psi(\zeta)}\mu(d\zeta).$$

With this definition we have obviously

$$(\varphi, \psi) = \overline{(\psi, \varphi)},$$

$$(\varphi_1 + \varphi_2, \psi) = (\varphi_1, \psi) + (\varphi_2, \psi),$$

$$(\alpha\varphi, \psi) = \alpha(\varphi, \psi) \quad \text{for every complex number } \alpha,$$

$$(\varphi, \varphi) = \int_{\Omega} |\varphi(\zeta)|^2 \mu(d\zeta) \geqq 0.$$

We see easily by the definition of the integral that

$$\int_{\Omega} |\varphi(\zeta)|^2 \mu(d\zeta) = 0$$

if and only if $\mu(\{\zeta : \varphi(\zeta) \neq 0\}) = 0$. Therefore for every $\varphi \in L_2$, denoting by C_φ the class of all functions $\psi \in L_2$ for which

$$\mu(\{\zeta : \varphi(\zeta) \neq \psi(\zeta)\}) = 0,$$

we see easily that $\varphi_1 \in C_\varphi$, $\psi_1 \in C_\psi$ implies

$$\alpha\varphi_1 + \beta\psi_1 \in C_{\alpha\varphi + \beta\psi}$$

for every complex numbers α, β. Thus, denoting by R the totality of such classes C_φ $(\varphi \in L_2)$, we obtain a linear space R, defining

$$\alpha C_\varphi + \beta C_\psi = C_{\alpha\varphi + \beta\psi}.$$

Furthermore we conclude easily that

$$(\varphi_1, \psi_1) = (\varphi, \psi) \quad \text{for every } \varphi_1 \in C_\varphi,\ \psi_1 \in C_\psi .$$

Thus we define (C_φ, C_ψ) for $\varphi, \psi \in L_2$ to mean

$$(C_\varphi, C_\psi) = (\varphi, \psi).$$

With this definition, (C_φ, C_ψ) $(\varphi, \psi \in L_2)$ is an inner product on R. In fact $(C_\varphi, C_\varphi) = 0$ implies

$$\mu(\{\zeta : \varphi(\zeta) \neq 0\}) = 0,$$

and hence $C_\varphi = 0$. Thus R is an inner product space. This inner product space R is called the (L_2)-space by μ.

Theorem 48.1. *The (L_2)-space by a measure $\mu(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ is complete, that is, a Hilbert space.*

Proof. For a sequence $\varphi_\nu \in L_2$ $(\nu = 1, 2, \ldots)$ if

$$\lim_{\nu, \rho \to \infty} \| C_{\varphi_\nu} - C_{\varphi_\rho} \| = 0,$$

then we can find a subsequence ρ_ν $(\nu = 1, 2, \ldots)$ such that

$$\| C_{\varphi_{\rho_{\nu+1}}} - C_{\varphi_{\rho_\nu}} \| \leqq \frac{1}{2^\nu} \qquad (\nu = 1, 2, \ldots).$$

Then, putting

$$\psi_\nu(\zeta) = \sum_{\sigma=1}^{\nu} |\varphi_{\rho_{\sigma+1}}(\zeta) - \varphi_{\rho_\sigma}(\zeta)| \qquad (\nu = 1, 2, \ldots),$$

we have obviously $0 \leqq \psi_1(\zeta) \leqq \psi_2(\zeta) \leqq \cdots$ for every $\zeta \in \Omega$,

$$\| C_{\psi_\nu} \| \leqq \sum_{\sigma=1}^{\nu} \| C_{\varphi_{\rho_{\sigma+1}}} - C_{\varphi_{\rho_\sigma}} \| \leqq \sum_{\sigma=1}^{\nu} \frac{1}{2^\sigma} = 1 - \frac{1}{2^{\nu+1}},$$

$\psi_\nu \in L_2$ for every $\nu = 1, 2, \ldots$, because $\| C_\varphi \| = \| C_{|\varphi|} \|$ for every $\varphi \in L_2$. Thus we have by definition

$$\lim_{\nu \to \infty} \int_\Omega |\psi_\nu(\zeta)|^2 \, \mu(d\zeta) < +\infty.$$

By virtue of Lemma 4, we can find hence a set $\mathfrak{E}_0 \in \mathfrak{M}$ such that

$$\mu(\mathfrak{E}_0) = \mu(\Omega),$$

$$\lim_{\nu \to \infty} \psi_\nu(\zeta) < +\infty \qquad \text{for every } \zeta \in \mathfrak{E}_0.$$

Then $\varphi_{\rho_\sigma}(\zeta)$ $(\sigma = 1, 2, \ldots)$ is convergent for every $\zeta \in \mathfrak{E}_0$, because

$$|\varphi_{\rho_\nu}(\zeta) - \varphi_{\rho_\sigma}(\zeta)| \leqq \psi_{\nu-1}(\zeta) - \psi_{\sigma-1}(\zeta) \qquad \text{for } \nu > \sigma.$$

Thus, putting

$$\varphi(\zeta) = \lim_{\nu \to \infty} \varphi_{\rho_\nu}(\zeta) \qquad \text{for } \zeta \in \mathfrak{E}_0,$$

we have for every $\zeta \in \mathfrak{E}_0$ and $\nu = 1, 2, \ldots$

$$|\varphi(\zeta) - \varphi_{\rho_\nu}(\zeta)| \leqq \sum_{\sigma=\nu}^{\infty} |\varphi_{\rho_{\sigma+1}}(\zeta) - \varphi_{\rho_\sigma}(\zeta)|.$$

From this relation we conclude by Lemma 1

$$\int_{\mathfrak{E}_0} |\varphi(\zeta) - \varphi_{\rho_\nu}(\zeta)|^2 \, \mu(d\zeta) \leqq \lim_{\lambda \to \infty} \int_{\mathfrak{E}_0} \Big\{ \sum_{\sigma=\nu}^{\lambda} |\varphi_{\rho_{\sigma+1}}(\zeta) - \varphi_{\rho_\sigma}(\zeta)| \Big\}^2 \mu(d\zeta)$$

$$\leqq \lim_{\lambda\to\infty} \| \sum_{\sigma=\nu}^{\lambda} C_{|\varphi_{\rho_{\sigma+1}} - \varphi_{\rho_\sigma}|} \|^2$$
$$\leqq \Big(\sum_{\sigma=\nu}^{\infty} \| C_{\varphi_{\rho_{\sigma+1}}} - C_{\varphi_{\rho_\sigma}} \|\Big)^2 \leqq \Big(\frac{1}{2^{\nu-1}}\Big)^2 .$$

Thus, putting $\varphi(\zeta) = 0$ for $\zeta \in \mathfrak{E}_0'$, we obtain $\varphi \in L_2$ and

$$\| C_\varphi - C_{\varphi_{\rho_\nu}} \| \leqq \frac{1}{2^{\nu-1}} \qquad \text{for every } \nu = 1, 2, \ldots .$$

Therefore we conclude $\lim_{\nu\to\infty} \| C_\varphi - C_{\varphi_{\rho_\nu}} \| = 0$.

Let R be the (L_2)-space by a measure $\mu(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$. For the characteristic function $\chi_{\mathfrak{E}}$ of a set $\mathfrak{E} \in \mathfrak{M}$, putting

$$E(\mathfrak{E}) C_\varphi = C_{\chi_{\mathfrak{E}}\varphi} \qquad (\varphi \in L_2),$$

we obtain a projection operator $E(\mathfrak{E})$ on R. In fact, we have for every $\varphi, \psi \in L_2$

$$(E(\mathfrak{E})C_\varphi, C_\psi) = \int_\Omega \chi_{\mathfrak{E}}(\zeta)\varphi(\zeta)\overline{\psi(\zeta)}\,\mu(d\zeta)$$
$$= \int_\Omega \varphi(\zeta)\overline{\chi_{\mathfrak{E}}(\zeta)\psi(\zeta)}\,\mu(d\zeta) = (C_\varphi, E(\mathfrak{E})C_\psi),$$

and it is obvious that $E(\mathfrak{E})E(\mathfrak{E}) = E(\mathfrak{E})$. Therefore $E(\mathfrak{E})$ is a projection operator by Theorem 11.8.

Furthermore $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ is a spectrality on $\mathfrak{M}$. In fact, we have obviously for every $\varphi \in L_2$

$$E(\mathfrak{E})E(\mathfrak{F})C_\varphi = C_{\chi_{\mathfrak{E}}\chi_{\mathfrak{F}}\varphi} = C_{\chi_{\mathfrak{E}\mathfrak{F}}\varphi} = E(\mathfrak{E}\mathfrak{F})C_\varphi,$$
$$E(\mathfrak{E})C_\varphi + E(\mathfrak{E}')C_\varphi = C_{(\chi_{\mathfrak{E}}+\chi_{\mathfrak{E}'})\varphi} = C_\varphi.$$

Thus we have by Theorem 12.6

$$E(\mathfrak{E}\dotplus\mathfrak{F}) = 1 - E(\mathfrak{E}'\mathfrak{F}') = 1 - E(\mathfrak{E}')E(\mathfrak{F}')$$
$$= E(\mathfrak{E}) + E(\mathfrak{F}) - E(\mathfrak{E})E(\mathfrak{F}) = E(\mathfrak{E}) \cup E(\mathfrak{F}).$$

If $\mathfrak{E}_1 \supset \mathfrak{E}_2 \supset \ldots$, $\prod_{\nu=1}^{\infty} \mathfrak{E}_\nu = 0$, $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$, then we have for every $\varphi \in L_2$

$$\| E(\mathfrak{E}_\nu) C_\varphi \|^2 = \int_{\mathfrak{E}_\nu} |\varphi(\zeta)|^2 \mu(d\zeta),$$

and hence by Lemma 3

$$\lim_{\nu\to\infty} E(\mathfrak{E}_\nu) C_\varphi = 0 \qquad \text{for every } \varphi \in L_2.$$

Thus we have $E(\mathfrak{E}_\nu) \downarrow_{\nu=1}^{\infty} 0$. Accordingly we conclude easily for every sequence $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$

$$E\Big(\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu\Big) = \lim_{\sigma\to\infty} E\Big(\sum_{\nu=1}^{\sigma} \mathfrak{E}_\nu\Big) = \bigcup_{\nu=1}^{\infty} E(\mathfrak{E}_\nu).$$

Therefore $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ is a spectrality by definition. This spectrality E is called the <u>spectrality</u> of the (L_2)-space R.

For the constant 1 on Ω we have obviously

$$\| E(\mathfrak{E}) C_1 \|^2 = \int_{\Omega} \chi_{\mathfrak{E}}(\zeta) \mu(d\zeta) = \int_{\mathfrak{E}} \mu(d\zeta) = \mu(\mathfrak{E}).$$

Thus $E(\mathfrak{E}) = 0$ implies $\mu(\mathfrak{E}) = 0$. Conversely if $\mu(\mathfrak{E}) = 0$, then we have for every $\varphi \in L_2$

$$\| E(\mathfrak{E}) C_\varphi \|^2 = \int_{\Omega} |\chi_{\mathfrak{E}}(\zeta)\varphi(\zeta)|^2 \mu(d\zeta)$$
$$= \int_{\mathfrak{E}} |\varphi(\zeta)|^2 \mu(d\zeta) = 0,$$

and hence $E(\mathfrak{E}) = 0$. Therefore we can state

Theorem 48.2. *For the spectrality* $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ *of the* (L_2)*-space by a measure* $\mu(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ *we have* $E(\mathfrak{E}) = 0$ *if and only if* $\mu(\mathfrak{E}) = 0$.

Let φ be a bounded $(\mathfrak{M})$-measurable function on Ω. For any $\varepsilon > 0$ we can find obviously a finite number of sets $0 \neq \mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots, \varkappa)$ such that

$$\Omega = \sum_{\nu=1}^{\varkappa} \mathfrak{E}_\nu, \qquad \mathfrak{E}_\nu \mathfrak{E}_\rho = 0 \quad \text{for} \quad \nu \neq \rho,$$
$$\sup_{\xi, \eta \in \mathfrak{E}_\nu} |\varphi(\xi) - \varphi(\eta)| \leq \varepsilon \qquad (\nu = 1, 2, \ldots, \varkappa).$$

For such $\mathfrak{E}_\nu \in \mathfrak{M}$ and for arbitrary $\zeta_\nu \in \mathfrak{E}_\nu$ $(\nu = 1, 2, \ldots, \varkappa)$ we have

$$\int_{\Omega} |\varphi(\zeta) - \sum_{\nu=1}^{\varkappa} \varphi(\zeta_\nu) \chi_{\mathfrak{E}_\nu}(\zeta)|^2 \mu(d\zeta)$$
$$= \sum_{\nu=1}^{\varkappa} \int_{\mathfrak{E}_\nu} |\varphi(\zeta) - \varphi(\zeta_\nu)|^2 \mu(d\zeta) \leq \varepsilon^2 \mu(\Omega).$$

Since $E(\mathfrak{E}) C_1 = C_{\chi_{\mathfrak{E}}}$ for the constant 1 on Ω, we obtain hence

$$\| C_\varphi - \sum_{\nu=1}^{\varkappa} \varphi(\zeta_\nu) E(\mathfrak{E}_\nu) C_1 \| \leq \varepsilon \sqrt{\mu(\Omega)}.$$

Thus for the spectral ring $\mathfrak{R}$ of E we have $C_\varphi \in [C_1]\mathfrak{R}$ if φ is bounded. For an arbitrary $\varphi \in L_2$ we can find a sequence $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ such that $\mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \ldots$, $\sum_{\nu=1}^{\infty} \mathfrak{E}_\nu = \Omega$, and φ is bounded in $\mathfrak{E}_\nu$ for every $\nu = 1, 2, \ldots$. For such $\mathfrak{E}_\nu \in \mathfrak{M}$ $(\nu = 1, 2, \ldots)$ we obtain by Lemma 3

$$\lim_{\nu \to \infty} \int_{\Omega} |\varphi(\zeta) - \chi_{\mathfrak{E}_\nu}(\zeta)\varphi(\zeta)|^2 \mu(d\zeta)$$
$$= \lim_{\nu \to \infty} \int_{\mathfrak{E}_\nu'} |\varphi(\zeta)|^2 \mu(d\zeta) = 0.$$

and hence $\lim_{\nu \to \infty} \| C_\varphi - C_{\chi_{\mathfrak{E}_\nu}\varphi} \| = 0$. Here $\chi_{\mathfrak{E}_\nu}\varphi$ is bounded

in Ω for every $\nu = 1, 2, \ldots$. Therefore we conclude by definition $[C_1] = 1$.

Thus for every $\chi \in R$ we have by Theorem 19.4

$$[\chi] = C_{[\chi]}[C_1] = C_{[\chi]} \in \mathfrak{R},$$

and hence $\mathfrak{R}$ is simple by Theorem 20.3. Furthermore we see easily by Theorem 14.3

$$\mathfrak{R} = \{ E(\mathfrak{X}) : \mathfrak{X} \in \mathfrak{M} \}.$$

Therefore for every $\varphi \in L_2$ we can find $\mathfrak{X}_0 \in \mathfrak{M}$ for which

$$[C_\varphi] = E(\mathfrak{X}_0),$$

and hence $E(\mathfrak{X}_0)C_\varphi = C_\varphi$ by §14(3). On the other hand, putting

$$\mathfrak{X}_\varphi = \{ \zeta : \varphi(\zeta) \neq 0 \},$$

we have obviously

$$E(\mathfrak{X}_\varphi)C_\varphi = C_{\chi_{\mathfrak{X}_\varphi}\varphi} = C_\varphi,$$

and hence $[C_\varphi] = E(\mathfrak{X}_\varphi)[C_\varphi]$ by §14(6'). Therefore we have

$$E(\mathfrak{X}_0) = E(\mathfrak{X}_0)E(\mathfrak{X}_\varphi) = E(\mathfrak{X}_0\mathfrak{X}_\varphi) \leqq E(\mathfrak{X}_\varphi)$$

and further

$$\int_{\mathfrak{X}_\varphi - \mathfrak{X}_0\mathfrak{X}_\varphi} |\varphi(\zeta)|^2 \mu(d\zeta) = \| E(\mathfrak{X}_\varphi)C_\varphi - E(\mathfrak{X}_0\mathfrak{X}_\varphi)C_\varphi \|^2 = 0.$$

Since $|\varphi(\zeta)| > 0$ for every $\zeta \in \mathfrak{X}_\varphi - \mathfrak{X}_0\mathfrak{X}_\varphi$, we conclude hence

$$\mu(\mathfrak{X}_\varphi - \mathfrak{X}_0\mathfrak{X}_\varphi) = 0,$$

and consequently $E(\mathfrak{X}_\varphi - \mathfrak{X}_0\mathfrak{X}_\varphi) = 0$ by Theorem 48.2. Thus we obtain $E(\mathfrak{X}_\varphi) = E(\mathfrak{X}_0\mathfrak{X}_\varphi) = E(\mathfrak{X}_0)$. Therefore we can state

Theorem 48.3. *Let $E(\mathfrak{X})$ $(\mathfrak{X} \in \mathfrak{M})$ be the spectrality of the (L_2)-space R by a measure $\mu(\mathfrak{X})$ $(\mathfrak{X} \in \mathfrak{M})$. The spectral ring $\mathfrak{R}$ of E is simple, and*

$$\mathfrak{R} = \{ E(\mathfrak{X}) : \mathfrak{X} \in \mathfrak{M} \},$$

$$[C_\varphi] = E(\{ \zeta : \varphi(\zeta) \neq 0 \}) \quad \text{for every } \varphi \in L_2.$$

§49 Regular σ-ideals

Let $\mathfrak{M}$ be a totally additive set class in an abstract space Ω. For a measure $\mu(\mathfrak{X})$ $(\mathfrak{X} \in \mathfrak{M})$ on $\mathfrak{M}$, putting

$$\mathfrak{J}_0 = \{ \mathfrak{X} : \mu(\mathfrak{X}') = 0 \},$$

we see easily that $\mathfrak{J}_0$ is a σ-ideal. Furthermore $\mathfrak{J}_0$ is separable. In fact, for a system $\mathfrak{X}_\lambda \in \mathfrak{M}$ $(\lambda \in \Lambda)$, if $\mathfrak{X}_\lambda \mathfrak{X}_\rho = 0$ for $\lambda \neq \rho$,

then we have

$$\sum_{\lambda \in \Lambda} \mu(\mathfrak{E}_\lambda) \leqq \mu(\Omega) < +\infty,$$

and hence $\mu(\mathfrak{E}_\lambda) = 0$ except for at most countable elements of Λ. Furthermore $\mu(\mathfrak{E}_\lambda) = 0$ implies $\mathfrak{E}_\lambda' \in \mathfrak{F}_0$. Therefore $\mathfrak{F}_0$ is separable by definition.

A σ-ideal $\mathfrak{F}$ in $\mathfrak{M}$ is said to be <u>regular</u>, if we can find a measure $\mu(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ for which we have

$$\mathfrak{F} = \{\mathfrak{E} : \mu(\mathfrak{E}') = 0\}.$$

With this definition we have hence

<u>Theorem</u> 49.1. <u>Every regular σ-ideal is separable.</u>

<u>Theorem</u> 49.2. <u>If a σ-ideal $\mathfrak{F}_0$ is regular, then every σ-ideal $\mathfrak{F} \supset \mathfrak{F}_0$ also is regular.</u>

<u>Proof</u>. If a σ-ideal $\mathfrak{F}_0$ is regular, then there exists by definition a measure μ_0 on $\mathfrak{M}$ such that

$$\mathfrak{F}_0 = \{\mathfrak{E} : \mu_0(\mathfrak{E}') = 0\}.$$

Since $\mathfrak{F}_0$ is separable by Theorem 49.1, for any σ-ideal $\mathfrak{F} \supset \mathfrak{F}_0$ we can find by Theorem 45.1 a set $\mathfrak{E}_0 \in \mathfrak{M}$ for which $\mathfrak{F} = \mathfrak{F}_0^{\mathfrak{E}_0}$. For such $\mathfrak{E}_0 \in \mathfrak{M}$, putting

$$\mu(\mathfrak{E}) = \mu_0(\mathfrak{E}_0\mathfrak{E}) \qquad (\mathfrak{E} \in \mathfrak{M}),$$

we see easily that μ also is a measure on $\mathfrak{M}$. Furthermore we have for every $\mathfrak{E} \in \mathfrak{M}$

$$\mu_0((\mathfrak{E} + \mathfrak{E}_0')') = \mu_0(\mathfrak{E}_0\mathfrak{E}') = \mu(\mathfrak{E}').$$

Thus we have $\mu(\mathfrak{E}') = 0$ if and only if $\mu_0((\mathfrak{E} + \mathfrak{E}_0')') = 0$, and hence by definition

$$\mathfrak{F}_0^{\mathfrak{E}_0} = \{\mathfrak{E} : \mu(\mathfrak{E}') = 0\}.$$

Therefore $\mathfrak{F}$ is regular by definition.

<u>Theorem</u> 49.3. <u>For every sequence of regular σ-ideals $\mathfrak{F}_\nu$ $(\nu = 1, 2, \ldots)$ the intersection $\prod_{\nu=1}^{\infty} \mathfrak{F}_\nu$ also is a regular σ-ideal.</u>

<u>Proof</u>. By assumption we can find a sequence of measures μ_ν $(\nu = 1, 2, \ldots)$ on $\mathfrak{M}$ such that

$$\mathfrak{F}_\nu = \{\mathfrak{E} : \mu_\nu(\mathfrak{E}') = 0\} \qquad (\nu = 1, 2, \ldots).$$

For such μ_ν $(\nu = 1, 2, \ldots)$ we can find a sequence of positive numbers α_ν $(\nu = 1, 2, \ldots)$ for which we have

$$\sum_{\nu=1}^{\infty} \alpha_\nu \mu_\nu(\Omega) < +\infty .$$

For such $\alpha_\nu > 0$ $(\nu = 1, 2, \ldots)$, putting

$$\mu(\mathfrak{E}) = \sum_{\nu=1}^{\infty} \alpha_\nu \mu_\nu(\mathfrak{E}) \qquad (\mathfrak{E} \in \mathfrak{M}),$$

we see easily that μ is a measure on $\mathfrak{M}$. Furthermore it is obvious that $\mu(\mathfrak{E}) = 0$ if and only if $\mu_\nu(\mathfrak{E}) = 0$ for all $\nu = 1, 2, \ldots$. Therefore we have

$$\prod_{\nu=1}^{\infty} \mathfrak{P}_\nu = \prod_{\nu=1}^{\infty} \{ \mathfrak{E} : \mu_\nu(\mathfrak{E}') = 0 \} = \{ \mathfrak{E} : \mu(\mathfrak{E}') = 0 \},$$

and hence $\prod_{\nu=1}^{\infty} \mathfrak{P}_\nu$ is a regular σ-ideal by definition.

The totality of regular σ-ideals in $\mathfrak{M}$ is called the regular property of $\mathfrak{M}$ and denoted by $\mathcal{P}_{\mathfrak{M}}^{*}$ in the sequel. It is evident by Theorem 49.1 that $\mathcal{P}_{\mathfrak{M}}^{*} \subset \mathcal{P}_{\mathfrak{M}}$ for the property $\mathcal{P}_{\mathfrak{M}}$ of $\mathfrak{M}$.

Let $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ be a spectrality on $\mathfrak{M}$ in a Hilbert space $\mathfrak{R}$. For an element $a \in \mathfrak{R}$, if $C_{[a]} \leqq E(\mathfrak{E})$, then we have naturally $E(\mathfrak{E}') C_{[a]} = 0$, and hence $E(\mathfrak{E}')a = 0$ by §14(19). Conversely if $E(\mathfrak{E}')a = 0$, then we have $E(\mathfrak{E}') C_{[a]} = 0$ by §14(19), and hence $C_{[a]} \leqq E(\mathfrak{E})$. Since $\| E(\mathfrak{E})a \|^2$ $(\mathfrak{E} \in \mathfrak{M})$ is obviously a measure on $\mathfrak{M}$, we obtain therefore by Theorem 45.4

Theorem 49.4. For every $x \in \mathfrak{R}$, putting

$$\mathfrak{P} = \{ \mathfrak{E} : C_{[x]} \leqq E(\mathfrak{E}) \},$$

we have $\mathfrak{P} \in \mathcal{P}_{\mathfrak{M}}^{*}$ and $E(\mathfrak{P}) = C_{[x]}$ for the derivative $E(\mathfrak{P})$ of a spectrality E on $\mathfrak{M}$.

Furthermore we obtain hence by Theorem 45.5

Theorem 49.5. For any separable σ-ideal $\mathfrak{P}$ we can find a regular σ-ideal $\mathfrak{q}$ such that $\mathfrak{q} \supset \mathfrak{P}$ and $E(\mathfrak{q}) = E(\mathfrak{P})$.

Let $m(\mathfrak{P})$ $(\mathfrak{P} \in \mathcal{P}_{\mathfrak{M}})$ be the multiplicity function of a spectrality E on $\mathfrak{M}$. Denoting by $m_{\mathfrak{R}}(P)$ $(P \in \mathfrak{R})$ the minimal multiplicity of the spectral ring $\mathfrak{R}$ of E, we have by the definition §46 (1)

$$m(\mathfrak{P}) = m_{\mathfrak{R}}(E(\mathfrak{P})) \qquad (\mathfrak{P} \in \mathcal{P}_{\mathfrak{M}}).$$

For any $\mathfrak{P} \in \mathcal{P}_{\mathfrak{M}}$ we can find by Theorem 49.5 $\mathfrak{q} \in \mathcal{P}_{\mathfrak{M}}^{*}$ for which $E(\mathfrak{q}) = E(\mathfrak{P})$, and hence $m(\mathfrak{P}) = m(\mathfrak{q})$. Thus for two

spectralities E and F on $\mathfrak{M}$, denoting by m_E the multiplicity function of E and by m_F that of F, if

$$m_E(\mathfrak{P}) = m_F(\mathfrak{P}) \quad \text{for every } \mathfrak{P} \in \mathcal{P}_{\mathfrak{M}}^{\infty},$$

then we also have the same for every $\mathfrak{P} \in \mathcal{P}_{\mathfrak{M}}$. Therefore we obtain by Theorem 46.3

Theorem 49.6. *For two spectralities $E(\mathfrak{X})$ and $F(\mathfrak{X})$ ($\mathfrak{X} \in \mathfrak{M}$) there exists a unitary operator U such that*

$$F(\mathfrak{X}) = U E(\mathfrak{X}) U^* \qquad (\mathfrak{X} \in \mathfrak{M}),$$

if and only if $m_E(\mathfrak{P}) = m_F(\mathfrak{P})$ for every $\mathfrak{P} \in \mathcal{P}_{\mathfrak{M}}^*$, denoting by m_E the multiplicity function of E and by m_F that of F.

Let $m(\mathfrak{P})$ ($\mathfrak{P} \in \mathcal{P}_{\mathfrak{M}}$) the multiplicity function of a spectrality $E(\mathfrak{X})$ ($\mathfrak{X} \in \mathfrak{M}$). $\mathfrak{M}$ is obviously itself a regular σ-ideal containing 0, and hence we have by the definition §45(11) $E(\mathfrak{M}) = 0$ as the derivative of E, considering $\mathfrak{M}$ a regular σ-ideal. Thus we have by the definition §46(1)

(1) $\quad m(\mathfrak{M}) = 0$, considering $\mathfrak{M} \in \mathcal{P}_{\mathfrak{M}}^*$.

If $E(\mathfrak{P}) = 0$, $\mathfrak{P} \subset \mathfrak{Q}$, then we have $E(\mathfrak{Q}) = 0$ by §45(15). Thus we have by the definition §46(1)

(2) $\quad m(\mathfrak{P}) = 0$, $\mathfrak{P} \subset \mathfrak{Q}$ implies $m(\mathfrak{Q}) = 0$.

For a sequence $\mathfrak{P}_\nu \in \mathcal{P}_{\mathfrak{M}}$ ($\nu = 1, 2, \ldots$) we have by §45(14)

$$E\left(\prod_{\nu=1}^{\infty} \mathfrak{P}_\nu\right) = \bigcup_{\nu=1}^{\infty} E(\mathfrak{P}_\nu).$$

Thus we have by definition

(3) $\quad m(\mathfrak{P}_\nu) = 0$ ($\nu = 1, 2, \ldots$) implies $m\left(\prod_{\nu=1}^{\infty} \mathfrak{P}_\nu\right) = 0$.

Furthermore we obtain by the formula §44(4)

(4) $\quad m\left(\prod_{\nu=1}^{\infty} \mathfrak{P}_\nu\right) = \operatorname*{Min}_{m(\mathfrak{P}_\nu) \geq 1} m(\mathfrak{P}_\nu)$ if $\sup_{\nu \geq 1} m(\mathfrak{P}_\nu) \geq 1$.

By virtue of Theorem 49.4, there is a regular σ-ideal $\mathfrak{P}$ for which we have $E(\mathfrak{P}) \neq 0$. Thus we have

(5) $\quad \sup_{\mathfrak{P} \in \mathcal{P}_{\mathfrak{M}}^a} m(\mathfrak{P}) \geq 1.$

In the sequel we shall prove that for any cardinal valued function $m(\mathfrak{P})$ ($\mathfrak{P} \in \mathcal{P}_{\mathfrak{M}}^a$) subject to the conditions (1), (2), (3), (4), and

(5), we can find a Hilbert space R and a spectrality $E(\mathfrak{E})$ ($\mathfrak{E} \in \mathfrak{M}$) in R such that the multiplicity function of E coincides with $\mathfrak{m}$.

For this purpose we must prove the following Lemmas about σ-ideals.

Lemma 1. $\mathfrak{J}^{\mathfrak{E}+\mathfrak{F}} = \mathfrak{J}^{\mathfrak{E}}$ if $\mathfrak{F}' \in \mathfrak{J}$.

Proof. If $\mathfrak{F}_0' \in \mathfrak{J}$, then $\mathfrak{E} + \mathfrak{E}_0' \in \mathfrak{J}$ implies

$$\mathfrak{E} + \mathfrak{E}_0'\mathfrak{F}_0' \supset \mathfrak{F}_0'(\mathfrak{E} + \mathfrak{E}_0') \in \mathfrak{J},$$

and hence $\mathfrak{E} + (\mathfrak{E}_0 + \mathfrak{F}_0)' \in \mathfrak{J}$. Conversely $\mathfrak{E} + (\mathfrak{E}_0 + \mathfrak{F}_0)' \in \mathfrak{J}$ implies $\mathfrak{E} + \mathfrak{E}_0' \in \mathfrak{J}$, because $\mathfrak{E}_0' \supset (\mathfrak{E}_0 + \mathfrak{F}_0)'$. Thus we have by definition

$$\mathfrak{J}^{\mathfrak{E}_0 + \mathfrak{F}_0} = \mathfrak{J}^{\mathfrak{E}_0} \quad \text{if} \quad \mathfrak{F}_0' \in \mathfrak{J}.$$

Lemma 2. $\mathfrak{J}^{\mathfrak{E}} \perp \mathfrak{J}^{\mathfrak{F}}$ if and only if $(\mathfrak{E}\mathfrak{F})' \in \mathfrak{J}$.

Proof. If $\mathfrak{J}^{\mathfrak{E}_0} \perp \mathfrak{J}^{\mathfrak{F}_0}$, then we can find by definition $\mathfrak{E} \in \mathfrak{J}^{\mathfrak{E}_0}$ and $\mathfrak{F} \in \mathfrak{J}^{\mathfrak{F}_0}$ for which $\mathfrak{E}\mathfrak{F} = 0$. For such $\mathfrak{E}$, $\mathfrak{F}$ we have

$$\mathfrak{E} + \mathfrak{E}_0' \in \mathfrak{J}, \quad \mathfrak{F} + \mathfrak{F}_0' \in \mathfrak{J}, \quad \mathfrak{E}\mathfrak{F} = 0,$$

and hence

$$\mathfrak{J} \ni (\mathfrak{E} + \mathfrak{E}_0')(\mathfrak{F} + \mathfrak{F}_0') \subset \mathfrak{E}_0' + \mathfrak{F}_0' = (\mathfrak{E}_0\mathfrak{F}_0)'.$$

Thus we have $\mathfrak{J} \ni (\mathfrak{E}_0\mathfrak{F}_0)'$ by definition.

Conversely if $(\mathfrak{E}_0\mathfrak{F}_0)' \in \mathfrak{J}$, then we have by Lemma 1

$$\mathfrak{J}^{\mathfrak{E}_0} = \mathfrak{J}^{\mathfrak{E}_0 - \mathfrak{E}_0\mathfrak{F}_0}, \qquad \mathfrak{J}^{\mathfrak{F}_0} = \mathfrak{J}^{\mathfrak{F}_0 - \mathfrak{E}_0\mathfrak{F}_0}$$

Since $(\mathfrak{E}_0 - \mathfrak{E}_0\mathfrak{F}_0)(\mathfrak{F}_0 - \mathfrak{E}_0\mathfrak{F}_0) = 0$, we have $\mathfrak{J}^{\mathfrak{E}_0} \perp \mathfrak{J}^{\mathfrak{F}_0}$ by definition, because $\mathfrak{E} \in \mathfrak{J}^{\mathfrak{E}}$ for every $\mathfrak{E} \in \mathfrak{M}$ by §45(6).

Lemma 3. For two separable σ-ideals $\mathfrak{J}$ and $\mathfrak{Q}$ we can find a set $\mathfrak{E} \in \mathfrak{M}$ such that $\mathfrak{J}^{\mathfrak{E}} \supset \mathfrak{Q}$ and $\mathfrak{J}^{\mathfrak{E}'} \perp \mathfrak{Q}$.

Proof. Putting $\mathfrak{J}_0 = \mathfrak{J}\mathfrak{Q}$, we can find by Theorems 45.1 and 45.3 two sets $\mathfrak{E}_0, \mathfrak{F}_0 \in \mathfrak{M}$ such that $\mathfrak{J} = \mathfrak{J}_0^{\mathfrak{E}_0}$, $\mathfrak{Q} = \mathfrak{J}_0^{\mathfrak{F}_0}$. Then we have by the formulas (9) and (4) in §45

$$\mathfrak{J}^{\mathfrak{F}_0} = \mathfrak{J}_0^{\mathfrak{E}_0\mathfrak{F}_0} = \mathfrak{Q}^{\mathfrak{E}_0} \supset \mathfrak{Q}$$

and furthermore $\mathfrak{J}^{\mathfrak{F}_0'} \perp \mathfrak{J}_0^{\mathfrak{F}_0} = \mathfrak{Q}$.

Lemma 4. $\mathfrak{J}^{\mathfrak{E}} = \mathfrak{M}$ if and only if $\mathfrak{E}' \in \mathfrak{J}$.

Proof. If $\mathfrak{J}^{\mathfrak{E}} = \mathfrak{M}$, then we have obviously $\mathfrak{E}' \in \mathfrak{J}^{\mathfrak{E}}$, and hence by definition $\mathfrak{E}' = \mathfrak{E}' + \mathfrak{E}' \in \mathfrak{J}$. Conversely if $\mathfrak{E}' \in \mathfrak{J}$, then we have by definition $0 \in \mathfrak{J}^{\mathfrak{E}}$, and hence $\mathfrak{J}^{\mathfrak{E}} = \mathfrak{M}$.

Lemma 5. For a separable σ-ideal $\mathfrak{J}$ and a system $\mathfrak{E}_\lambda \in \mathfrak{M}$

$(\lambda \in \Lambda)$, *if* $(\mathfrak{E}_\lambda \mathfrak{E}_\rho)' \in \mathfrak{P}$ *for* $\lambda \neq \rho$, *then we have* $\mathfrak{E}_\lambda' \in \mathfrak{P}$ *except for at most countable elements of* Λ.

Proof. By virtue of Maximal Theorem, we can find a maximal system $\mathfrak{F}_\lambda$ $(\lambda \in \Gamma)$ such that $\Gamma \subset \Lambda$, $\mathfrak{F}_\lambda \subset \mathfrak{E}_\lambda$ for every $\lambda \in \Gamma$, $\mathfrak{F}_\lambda \mathfrak{F}_\gamma = 0$ for $\lambda \neq \gamma$, and $\mathfrak{F}_\lambda' \bar{\in} \mathfrak{P}$ for every $\lambda \in \Gamma$. Since $\mathfrak{P}$ is separable by assumption, we see by definition that such Γ is composed of at most countable elements. Thus, putting $\mathfrak{E}_0 = \sum_{\lambda \in \Gamma} \mathfrak{E}_\lambda$, we have $\mathfrak{E}_0 \in \mathfrak{m}$. For each $\lambda_0 \bar{\in} \Gamma$, since $(\mathfrak{E}_\lambda \mathfrak{E}_{\lambda_0})' \in \mathfrak{P}$ for $\lambda \in \Gamma$ by assumption, we also have by definition

$$\mathfrak{P} \ni \prod_{\lambda \in \Gamma} (\mathfrak{E}_\lambda \mathfrak{E}_{\lambda_0})' = (\mathfrak{E}_0 \mathfrak{E}_{\lambda_0})'.$$

Thus, if $\mathfrak{E}_{\lambda_0}' \bar{\in} \mathfrak{P}$, then we have $(\mathfrak{E}_{\lambda_0} \mathfrak{E}_0')' \bar{\in} \mathfrak{P}$, because $(\mathfrak{E}_{\lambda_0} \mathfrak{E}_0')' \in \mathfrak{P}$ implies $\mathfrak{P} \ni (\mathfrak{E}_{\lambda_0} \mathfrak{E}_0')' (\mathfrak{E}_0 \mathfrak{E}_{\lambda_0})' = \mathfrak{E}_{\lambda_0}'$. Accordingly, putting $\mathfrak{F}_{\lambda_0} = \mathfrak{E}_{\lambda_0} \mathfrak{E}_0'$, we have $\mathfrak{F}_{\lambda_0} \subset \mathfrak{E}_{\lambda_0}$, $\mathfrak{F}_{\lambda_0}' \bar{\in} \mathfrak{P}$ and

$$\mathfrak{F}_{\lambda_0} \mathfrak{F}_\lambda \subset \mathfrak{E}_{\lambda_0} \mathfrak{E}_0' \mathfrak{E}_\lambda = 0 \qquad \text{for every } \lambda \in \Gamma,$$

contradicting the maximality of the system $\mathfrak{F}_\lambda$ $(\lambda \in \Gamma)$. Therefore we have $\mathfrak{E}_\lambda' \in \mathfrak{P}$ for $\lambda \bar{\in} \Gamma$.

Let $m(\mathfrak{P})$ $(\mathfrak{P} \in \mathcal{P}_{\mathfrak{m}}^*)$ be a cardinal valued function on the regular property $\mathcal{P}_{\mathfrak{m}}^*$ of $\mathfrak{m}$ satisfying the conditions (1)-(5). A regular σ-ideal $\mathfrak{P} \in \mathcal{P}_{\mathfrak{m}}^*$ is said to be of *uniform multiplicity* by m, if $m(\mathfrak{P}) \geqq 1$ and $\mathfrak{m} \neq \mathfrak{q} \supset \mathfrak{P}$ implies $m(\mathfrak{q}) = m(\mathfrak{P})$. We shall prove firstly that for any $\mathfrak{P}_0 \in \mathcal{P}_{\mathfrak{m}}^*$ with $m(\mathfrak{P}_0) \geqq 1$, we can find $\mathfrak{q} \supset \mathfrak{P}_0$ such that $m(\mathfrak{q}) = m(\mathfrak{P}_0)$ and $\mathfrak{q}$ is of uniform multiplicity. We suppose that $\mathfrak{P}_0$ is not of uniform multiplicity. Then there exists by definition $\mathfrak{m} \neq \mathfrak{q} \supset \mathfrak{P}_0$ such that $m(\mathfrak{q}) \neq m(\mathfrak{P}_0)$. For such $\mathfrak{q}$ we can find by Theorem 45.1 a set $\mathfrak{E}_0 \in \mathfrak{m}$ for which $\mathfrak{q} = \mathfrak{P}_0^{\mathfrak{E}_0}$, and hence $\mathfrak{E}_0' \bar{\in} \mathfrak{P}$ by Lemma 4. Therefore we can find by Maximal Theorem a maximal system $\mathfrak{E}_\lambda \in \mathfrak{m}$ $(\lambda \in \Lambda)$ such that $\mathfrak{E}_\lambda \mathfrak{E}_\rho = 0$ for $\lambda \neq \rho$, $\mathfrak{E}_\lambda' \bar{\in} \mathfrak{P}_0$, and

$$m(\mathfrak{P}_0^{\mathfrak{E}_\lambda}) \neq m(\mathfrak{P}_0) \qquad \text{for every } \lambda \in \Lambda.$$

Since $\mathfrak{P}_0$ is separable, such Λ is composed of at most countable elements. Furthermore we have

$$\sum_{\lambda \in \Lambda} \mathfrak{E}_\lambda \bar{\in} \mathfrak{P}_0,$$

because, if $\sum_{\lambda \in \Lambda} \mathfrak{E}_\lambda \in \mathfrak{P}_0$, then we have by the formulas (5) and (8) in

$$\mathfrak{f}_0 = \mathfrak{f}_0^{\sum_{\lambda\in\Lambda}\mathfrak{E}_\lambda} = \prod_{\lambda\in\Lambda}\mathfrak{f}_0^{\mathfrak{E}_\lambda}$$

but $m\left(\prod_{\lambda\in\Lambda}\mathfrak{f}_0^{\mathfrak{E}_\lambda}\right) \neq m(\mathfrak{f}_0)$ by the conditions (3) and (4). Thus, putting $\mathfrak{E}_0 = \left(\sum_{\lambda\in\Lambda}\mathfrak{E}_\lambda\right)$, we have $\mathfrak{E}_0' \in \mathfrak{f}_0$. Furthermore

$$m(\mathfrak{f}_0^{\mathfrak{E}}) = m(\mathfrak{f}_0) \quad \text{for } \mathfrak{E}_0 \supset \mathfrak{E} \in \mathfrak{m},\ \mathfrak{E}' \in \mathfrak{f}_0.$$

Because, if $m(\mathfrak{f}_0^{\mathfrak{E}}) \neq m(\mathfrak{f}_0)$ for some $\mathfrak{E} \subset \mathfrak{E}_0$ and $\mathfrak{E}' \in \mathfrak{f}_0$, then we have $\mathfrak{E}\mathfrak{E}_\lambda = 0$ for every $\lambda \in \Lambda$, contradicting the maximality of the system $\mathfrak{E}_\lambda (\lambda \in \Lambda)$. Therefore we see by Theorem 45.1 and the formula §45(9) that $\mathfrak{f}_0^{\mathfrak{E}_0}$ is of uniform multiplicity and

$$m(\mathfrak{f}_0^{\mathfrak{E}_0}) = m(\mathfrak{f}_0).$$

By virtue of Maximal Theorem, we can find hence by the condition (5) a maximal system $\mathfrak{f}_\lambda \in \mathcal{P}_m^*$ $(\lambda \in \Lambda)$ such that $\mathfrak{f}_\lambda \perp \mathfrak{f}_\rho$ for $\lambda \neq \rho$, and every $\mathfrak{f}_\lambda (\lambda \in \Lambda)$ is of uniform multiplicity. For such $\mathfrak{f}_\lambda \in \mathcal{P}_m^*$ $(\lambda \in \Lambda)$, if $\mathcal{P}_m^* \ni \mathfrak{q} \perp \mathfrak{f}_\lambda$ for every $\lambda \in \Lambda$, then we have $m(\mathfrak{q}) = 0$. Because, if $m(\mathfrak{q}) \geq 1$, then we can find $\mathfrak{q}_0 \supset \mathfrak{q}$ of uniform multiplicity, as proved just above, and for such $\mathfrak{q}_0$ we have obviously $\mathfrak{q}_0 \perp \mathfrak{f}_\lambda$ for every $\lambda \in \Lambda$, contradicting the maximality of the system $\mathfrak{f}_\lambda (\lambda \in \Lambda)$.

For an arbitrary $\mathfrak{q} \in \mathcal{P}_m^*$ we can find by Lemma 3 $\mathfrak{E}_\lambda \in \mathfrak{m}$ $(\lambda \in \Lambda)$ such that

$$\mathfrak{q}^{\mathfrak{E}_\lambda} \supset \mathfrak{f}_\lambda, \quad \mathfrak{q}^{\mathfrak{E}_\lambda'} \perp \mathfrak{f}_\lambda \quad \text{for every } \lambda \in \Lambda.$$

For such $\mathfrak{E}_\lambda \in \mathfrak{m}$ $(\lambda \in \Lambda)$ we have naturally $\mathfrak{q}^{\mathfrak{E}_\lambda} \perp \mathfrak{q}^{\mathfrak{E}_\rho}$ for $\lambda \neq \rho$, and hence by Lemma 2 $(\mathfrak{E}_\lambda\mathfrak{E}_\rho)' \in \mathfrak{q}$ for $\lambda \neq \rho$. Accordingly we have by Lemma 5 $\mathfrak{E}_\lambda' \in \mathfrak{q}$ except for at most countable elements of Λ. Thus, putting $\mathfrak{E} = \sum_{\mathfrak{E}_\lambda' \in \mathfrak{q}} \mathfrak{E}_\lambda$, we have $\mathfrak{E} \in \mathfrak{m}$ and by the formulas (8) and (5) in §45

$$\mathfrak{q} = \mathfrak{q}^{\mathfrak{E}}\mathfrak{q}^{\mathfrak{E}'} = \mathfrak{q}^{\mathfrak{E}'}\prod_{\mathfrak{E}_\lambda'\in\mathfrak{q}}\mathfrak{q}^{\mathfrak{E}_\lambda}.$$

On the other hand, for $\mathfrak{E}_\lambda' \in \mathfrak{q}$ we have by the formulas (4) and (5) in §45 $\mathfrak{q}^{\mathfrak{E}'} \supset \mathfrak{q}^{\mathfrak{E}_\lambda'} \perp \mathfrak{f}_\lambda$, and hence $\mathfrak{q}^{\mathfrak{E}'} \perp \mathfrak{f}_\lambda$. And for $\mathfrak{E}_\lambda' \in \mathfrak{q}$, since $\mathfrak{E} \supset \mathfrak{E}_\lambda$, we have by the formulas (4) and (9) in §45 $\mathfrak{q}^{\mathfrak{E}'} \supset \mathfrak{q}^{\mathfrak{E}_\lambda'} \perp \mathfrak{f}_\lambda$, and hence $\mathfrak{q}^{\mathfrak{E}'} \perp \mathfrak{f}_\lambda$. Thus we have

$$\mathfrak{q}^{\mathfrak{E}'} \perp \mathfrak{f}_\lambda \quad \text{for every } \lambda \in \Lambda,$$

and consequently $m(\mathfrak{q}^{\mathfrak{E}'}) = 0$, as proved just above. Furthermore for $\mathfrak{E}_\lambda' \in \mathfrak{q}$ we have by Lemma 4 $\mathfrak{m} \neq \mathfrak{q}^{\mathfrak{E}_\lambda} \supset \mathfrak{f}_\lambda$ and hence

$$m(\mathfrak{q}^{\mathfrak{E}_\lambda}) = m(\mathfrak{f}_\lambda),$$

because $\mathfrak{F}_\lambda$ is of uniform multiplicity. Therefore, if $\mathfrak{q} \perp \mathfrak{F}_\lambda$ not for all $\lambda \in \Lambda$, then we have by the condition (4)

$$(\#) \qquad m(\mathfrak{q}) = \min_{\mathfrak{q} \perp \mathfrak{F}_\lambda} m(\mathfrak{F}_\lambda),$$

adopting the notation $\mathfrak{q} \mathrel{I} \mathfrak{p}$ to mean that we have not $\mathfrak{q} \perp \mathfrak{p}$.

Since every $\mathfrak{F}_\lambda$ $(\lambda \in \Lambda)$ is a regular σ-ideal, we can find by definition a system of measures $\mu_\lambda(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ $(\lambda \in \Lambda)$ such that

$$\mathfrak{F}_\lambda = \{ \mathfrak{E} : \mu_\lambda(\mathfrak{E}') = 0 \} \qquad \text{for every } \lambda \in \Lambda.$$

Let R_λ be the (L_2)-space by this measure μ_λ and $F_\lambda(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ the spectrality of R_λ for every $\lambda \in \Lambda$. Let Δ be a set with density $\sum_{\lambda \in \Lambda} m(\mathfrak{F}_\lambda)$ and Δ_λ a subset of Δ with density $m(\mathfrak{F}_\lambda)$ for every $\lambda \in \Lambda$ such that $\Delta_\lambda \Delta_\rho = 0$ for $\lambda \neq \rho$ and $\Delta = \sum_{\lambda \in \Lambda} \Delta_\lambda$.

Denoting by R the totality of these systems $(x_\delta)_{\delta \in \Delta}$ subject to the condition: $x_\delta \in R_\lambda$ for $\delta \in \Delta_\lambda$ $(\lambda \in \Lambda)$ and $\sum_{\delta \in \Delta} \| x_\delta \|^2 < +\infty$, we see easily that R is a Hilbert space, if we define

$$\alpha (x_\delta)_{\delta \in \Delta} + \beta (y_\delta)_{\delta \in \Delta} = (\alpha x_\delta + \beta y_\delta)_{\delta \in \Delta}$$

for every complex numbers α, β and

$$((x_\delta)_{\delta \in \Delta}, (y_\delta)_{\delta \in \Delta}) = \sum_{\delta \in \Delta} (x_\delta, y_\delta).$$

For this Hilbert space R, putting

$$F_{\delta_0}^R(\mathfrak{E})(x_\delta)_{\delta \in \Delta}$$
$$= (y_\delta : y_{\delta_0} = F_{\lambda_0}(\mathfrak{E}) x_{\delta_0} \text{ for } \delta_0 \in \Delta_{\lambda_0} \text{ and } y_\delta = 0 \text{ for } \delta \neq \delta_0)_{\delta \in \Delta}$$

for every $\mathfrak{E} \in \mathfrak{M}$ and $\delta_0 \in \Delta$, we obtain obviously a projection operator $F_{\delta_0}^R(\mathfrak{E})$ on R and $F_{\delta_0}^R(\Omega)R$ is isometric to R_{λ_0} for $\delta_0 \in \Delta_{\lambda_0}$ such that $F_{\delta_0}^R(\mathfrak{E})R$ corresponds to $F_{\lambda_0}(\mathfrak{E})R_{\lambda_0}$ for every $\mathfrak{E} \in \mathfrak{M}$. Therefore we see by Theorem 48.3 that $F_{\delta_0}^R(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{M})$ is a spectrality in $F_{\delta_0}^R(\Omega)R$, and

$$\{ F_{\delta_0}^R(\mathfrak{E}) : \quad \mathfrak{E} \in \mathfrak{M} \}$$

is the spectral ring of $F_{\delta_0}^R$ and simple. Furthermore we see by Theorem 48.2 that for $\delta_0 \in \Delta_{\lambda_0}$ we have

$$\mathfrak{E} \in \mathfrak{F}_{\lambda_0} \quad \text{if and only if} \quad F_{\delta_0}^R(\mathfrak{E}') = 0.$$

Putting

$$E(\mathfrak{E}) = \bigcup_{\delta \in \Delta} F_\delta^R(\mathfrak{E}) \qquad (\mathfrak{E} \in \mathfrak{M}),$$

we see easily that E is a spectrality on $\mathfrak{M}$ in R. Concerning the derivative of this spectrality E, we have by the definition §45(11)

$$E(\mathfrak{F}_\lambda) = \bigcap_{\mathfrak{E} \in \mathfrak{F}_\lambda} E(\mathfrak{E}) \qquad (\lambda \in \Lambda).$$

For each $\mathfrak{F} \in \mathfrak{F}_\lambda$, we have $F_\delta^R(\mathfrak{F}') = 0$ for every $\delta \in \Delta_\lambda$, and hence $F_\delta^R(\Omega) = F_\delta^R(\mathfrak{F})$ for every $\delta \in \Delta_\lambda$. Therefore we have

$$E(\mathfrak{F}) \geq \bigcup_{\delta \in \Delta_\lambda} F_\delta^R(\Omega) \qquad \text{for every } \mathfrak{F} \in \mathfrak{F}_\lambda,$$

and consequently

$$E(\mathfrak{F}_\lambda) \geq \bigcup_{\delta \in \Delta_\lambda} F_\delta^R(\Omega).$$

On the other hand, for $\lambda \neq \rho \in \Lambda$ we have $\mathfrak{F}_\lambda \perp \mathfrak{F}_\rho$, and hence we can find by definition two sets $\mathfrak{F}_0$, $\mathfrak{F}_0 \in \mathfrak{M}$ such that

$$\mathfrak{F}_0 \in \mathfrak{F}_\lambda, \quad \mathfrak{F}_0 \in \mathfrak{F}_\rho, \quad \mathfrak{F}_0 \mathfrak{F}_0 = 0.$$

For such $\mathfrak{F}_0$ and $\mathfrak{F}_0$, we have obviously $\mathfrak{F}_0 \subset \mathfrak{F}_0'$ and hence $\mathfrak{F}_0' \in \mathfrak{F}_\rho$. Thus we have $F_\delta^R(\mathfrak{F}_0) = 0$ for $\delta \in \Delta_\rho$. Since

$$F_{\delta_1}^R(\Omega) F_{\delta_2}^R(\Omega) = 0 \qquad \text{for } \delta_1 \neq \delta_2,$$

we obtain hence $E(\mathfrak{F}_0) \bigcup_{\delta \in \Delta_\rho} F_\delta^R(\Omega) = 0$. Therefore we conclude

$$E(\mathfrak{F}_\lambda) \bigcup_{\delta \in \Delta_\rho} F_\delta^R(\Omega) = 0 \qquad \text{for } \lambda \neq \rho \in \Lambda,$$

and consequently $E(\mathfrak{F}_\lambda) \bigcup_{\lambda \neq \rho \in \Lambda} \bigcup_{\delta \in \Delta_\rho} F_\delta^R(\Omega) = 0$. Thus we obtain

$$E(\mathfrak{F}_\lambda) \leq 1 - \bigcup_{\lambda \neq \rho \in \Lambda} \bigcup_{\delta \in \Delta_\rho} F_\delta^R(\Omega) = \bigcup_{\delta \in \Delta_\lambda} F_\delta^R(\Omega),$$

because $\bigcup_{\lambda \in \Lambda} \bigcup_{\delta \in \Delta_\lambda} F_\delta^R(\Omega) = 1$. Therefore we have

$$E(\mathfrak{F}_\lambda) = \bigcup_{\delta \in \Delta_\lambda} F_\delta^R(\Omega).$$

Denoting by $\mathfrak{R}$ the spectrality of E, we obtain hence by Theorem 45.7

$$E(\mathfrak{F}_\lambda)\mathfrak{R} = \left\{ \bigcup_{\delta \in \Delta_\lambda} F_\delta^R(\mathfrak{F}) : \mathfrak{F} \in \mathfrak{M} \right\} \qquad (\lambda \in \Lambda).$$

Therefore $E(\mathfrak{F}_\lambda)\mathfrak{R}$ is of uniform multiplicity with multiplicity $m(\mathfrak{F}_\lambda)$. Thus, denoting by $m_\mathfrak{R}(\mathfrak{F})$ $(\mathfrak{F} \in \mathcal{P}_\mathfrak{M})$ the multiplicity function of E, we have

$$m_\mathfrak{R}(\mathfrak{F}_\lambda) = m(\mathfrak{F}_\lambda) \qquad \text{for every } \lambda \in \Lambda.$$

For a σ-ideal $\eta \in \mathcal{P}_\mathfrak{M}^*$, if $\eta \perp \mathfrak{F}_\lambda$, then we have by §45(13) $E(\mathfrak{F}_\lambda) E(\eta) = 0$. Thus, if $\eta \perp \mathfrak{F}_\lambda$ for all $\lambda \in \Lambda$, then we have

$$\bigcup_{\lambda \in \Lambda} E(\mathfrak{F}_\lambda) E(\eta) = 0.$$

Since $\bigcup_{\lambda \in \Lambda} E(\mathfrak{F}_\lambda) = \bigcup_{\lambda \in \Lambda} \bigcup_{\delta \in \Delta_\lambda} F_\delta^R(\Omega) = 1$, we obtain hence $E(\eta) = 0$, and consequently $m_\mathfrak{R}(\eta) = 0$ by definition.

If $\eta \not\perp \mathfrak{F}_\lambda$, then we can find by Lemma 3 a set $\mathfrak{F} \in \mathfrak{M}$ such that $\mathfrak{F}_\lambda^\mathfrak{F} \supset \eta$ and $\mathfrak{F}_\lambda^{\mathfrak{F}'} \perp \eta$. For such $\mathfrak{F} \in \mathfrak{M}$ we have $\mathfrak{F}' \notin \mathfrak{F}_\lambda$, because $\mathfrak{F}' \in \mathfrak{F}_\lambda$ implies by §45(5) $\mathfrak{F}_\lambda = \mathfrak{F}_\lambda^{\mathfrak{F}'} \perp \eta$, contradicting the assumption: $\eta \not\perp \mathfrak{F}_\lambda$. Thus we have

$$F_\delta^R(\mathfrak{E}) \neq 0 \qquad \text{for every } \delta \in \Delta_\lambda .$$

On the other hand we have by §45(12) for every $\delta \in \Delta_\lambda$

$$E(\mathfrak{f}_\lambda^{\mathfrak{E}}) = E(\mathfrak{E})E(\mathfrak{f}_\lambda) \geq F_\delta^R(\mathfrak{E})F_\delta^R(\Omega) = F_\delta^R(\mathfrak{E}).$$

Therefore we obtain $E(\mathfrak{f}_\lambda^{\mathfrak{E}}) \neq 0$. Since we have by §45(15)

$$E(\mathfrak{f}_\lambda)E(\mathfrak{q}) \geq E(\mathfrak{f}_\lambda)E(\mathfrak{f}_\lambda^{\mathfrak{E}}) = E(\mathfrak{f}_\lambda^{\mathfrak{E}}),$$

we conclude hence

$$E(\mathfrak{f}_\lambda)E(\mathfrak{q}) \neq 0 \qquad \text{for } \mathfrak{q} \bar{\perp} \mathfrak{f}_\lambda .$$

Furthermore we have

$$\bigcup_{\lambda\in\Lambda} E(\mathfrak{f}_\lambda)E(\mathfrak{q}) = \bigcup_{\lambda\in\Lambda}\bigcup_{\delta\in\Delta_\lambda} F_\delta^R(\Omega)E(\mathfrak{q}) = E(\mathfrak{q}),$$

and $E(\mathfrak{f}_\lambda)\mathcal{R}$ is of uniform multiplicity for every $\lambda \in \Lambda$. Therefore we obtain by (#) and the formula §44(4)

$$m_R(\mathfrak{q}) = \operatorname*{Min}_{\mathfrak{f}_\lambda \bar{\perp} \mathfrak{q}} m_R(\mathfrak{f}_\lambda) = \operatorname*{Min}_{\mathfrak{f}_\lambda \bar{\perp} \mathfrak{q}} m(\mathfrak{f}_\lambda) = m(\mathfrak{q})$$

if $\mathfrak{q} \perp \mathfrak{f}_\lambda$ not for all $\lambda \in \Lambda$.

Thus we have proved

Theorem 49.7. *If a cardinal valued function $m(\mathfrak{f})$ ($\mathfrak{f} \in \mathfrak{P}_m^*$) on the regular property $\mathfrak{P}_m^*$ of m satisfies the conditions* (1), (2), (3), (4), *and* (5), *then we can find a Hilbert space R and a spectrality $E(\mathfrak{E})$ ($\mathfrak{E} \in m$) in R such that the multiplicity function of E coincides with m.*

CHAPTER VI

ERGODIC THEOREMS

§50 Ergodic Theorem of Bounded Linear Operators

Let T be a bounded linear operator on a Hilbert space R and

$$\| T^{\nu} \| \leqq \gamma \qquad \text{for every } \nu = 1, 2, \ldots .$$

In the sequel we make use of the notation: $T^{0} = 1$ and

$$T_{\mu} = \sum_{\nu=0}^{\mu-1} T^{\nu} \qquad (\mu = 1, 2, \ldots).$$

Denoting by M the totality of those elements $x \in R$ for which

$$\frac{1}{\mu} T_{\mu} x \qquad (\mu = 1, 2, \ldots)$$

is convergent, we see easily that M is a linear manifold. Furthermore M is closed. In fact, if $\lim_{\nu \to \infty} x_{\nu} = x$, $x_{\nu} \in M$ $(\nu = 1, 2, \ldots)$, then we have for every $\mu = 1, 2, \ldots$

$$\| \frac{1}{\mu} T_{\mu} x_{\nu} - \frac{1}{\mu} T_{\mu} x \| \leqq \gamma \| x_{\nu} - x \|$$

and hence for every $\mu, \rho = 1, 2, \ldots$

$$\| \frac{1}{\mu} T_{\mu} x - \frac{1}{\rho} T_{\rho} x \| \leqq \| \frac{1}{\mu} T_{\mu} x_{\nu} - \frac{1}{\rho} T_{\rho} x_{\nu} \| + 2 \gamma \| x_{\nu} - x \| .$$

This relation yields for every $\nu = 1, 2, \ldots$

$$\overline{\lim_{\mu, \rho \to \infty}} \| \frac{1}{\mu} T_{\mu} x - \frac{1}{\rho} T_{\rho} x \| \leqq 2 \gamma \| x_{\nu} - x \| ,$$

and consequently

$$\lim_{\mu, \rho \to \infty} \| \frac{1}{\mu} T_{\mu} x - \frac{1}{\rho} T_{\rho} x \| = 0.$$

Thus $\frac{1}{\mu} T_{\mu} x$ $(\mu = 1, 2, \ldots)$ is convergent, that is, $x \in M$.

For every $x \in R$ and $\mu = 1, 2, \ldots$, we have

$$\lim_{\nu \to \infty} \frac{1}{\nu} T_{\nu} (x - T^{\mu} x) = 0,$$

because for $\nu > \mu$ we have

$$\begin{aligned} \frac{1}{\nu} T_{\nu} (1 - T^{\mu}) x &= \frac{1}{\nu} \Big(\sum_{\rho=0}^{\nu-1} T^{\rho} \Big) (1 - T^{\mu}) x \\ &= \frac{1}{\nu} \Big(\sum_{\rho=0}^{\mu-1} T^{\rho} x - \sum_{\rho=\nu}^{\nu+\mu-1} T^{\rho} x \Big), \end{aligned}$$

and hence

$$\| \frac{1}{\nu} T_{\nu} (1 - T^{\mu}) x \| \leqq \frac{2 \mu \gamma}{\nu} \| x \| \qquad \text{for } \nu > \mu .$$

Since $x - \frac{1}{\mu} T_\mu x = \frac{1}{\mu} \sum_{\rho=1}^{\mu} (x - T^{\rho-1}x)$, we obtain further

$$\lim_{\nu\to\infty} \frac{1}{\nu} T_\nu (x - \frac{1}{\mu} T_\mu x) = 0$$

for every $x \in R$ and $\mu = 1, 2, \ldots$.

Since $\| \frac{1}{\mu} T_\mu x \| \leqq \gamma \| x \|$ for every $\mu = 1, 2, \ldots$, we can find by Theorem 9.11 a subsequence $\mu_\nu (\nu = 1, 2, \ldots)$ for which

$$\frac{1}{\mu_\nu} T_{\mu_\nu} x \qquad (\nu = 1, 2, \ldots)$$

is weakly convergent. For such μ_ν ($= 1, 2, \ldots$), putting

$$y = \text{w-}\lim_{\nu\to\infty} \frac{1}{\mu_\nu} T_{\mu_\nu} x ,$$

we have by Theorem 10.9

$$Ty = \text{w-}\lim_{\nu\to\infty} T(\frac{1}{\mu_\nu} T_{\mu_\nu} x)$$

and hence

$$y - Ty = \text{w-}\lim_{\nu\to\infty} \frac{1}{\mu_\nu} T_{\mu_\nu}(x - Tx).$$

On the other hand we have

$$\lim_{\nu\to\infty} \frac{1}{\nu} T_\nu (x - Tx) = 0,$$

as proved just above. Thus we have $Ty = y$, and hence $y \in M$. Since

$$x - \frac{1}{\mu_\nu} T_{\mu_\nu} x \in M \qquad \text{for every } \nu = 1, 2, \ldots,$$

as proved just above, we also have

$$x - y = \text{w-}\lim_{\nu\to\infty} (x - \frac{1}{\mu_\nu} T_{\mu_\nu} x) \in M,$$

because M is a closed linear manifold. Thus we obtain

$$x = (x - y) + y \in M \qquad \text{for every } x \in R.$$

Therefore we have

Theorem 50.1. *For a bounded linear operator T on a Hilbert space R, if $\sup_{\nu \geq 1} \| T^\nu \| < +\infty$, then the sequence*

$$\frac{1}{\nu} \sum_{\mu=1}^{\nu} T^{\mu-1} x \qquad (\nu = 1, 2, \ldots)$$

is convergent for every $x \in R$.

This Theorem 50.1 is called Mean Ergodic Theorem of bounded linear operators.

§51 Ergodic Theorem of Dilatators

Let $\mathfrak{R}$ be a closed ring of projection operators containing 1 in a Hilbert space $\mathfrak{H}$. For a bounded dilatator T by $\mathfrak{R}$, we have by the formula §25(33)

$$[1 - T](1 - T) = 1 - T,$$

and hence

$$(1 - [1 - T])T = 1 - [1 - T].$$

Thus we have

$$(1 - [1 - T])T^{\nu} = 1 - [1 - T]$$

for every $\nu = 1, 2, \ldots$.

If $\|T\| \leqq 1$, then we have by Theorem 25.4

$$|T^{\nu}| \leqq 1 \qquad \text{for every } \nu = 1, 2, \ldots,$$

and hence we obtain by the formulas §25(40) for every $\mu = 1, 2, \ldots$

$$\begin{aligned} \left|[1 - T]\frac{1}{\mu}\sum_{\nu=0}^{\mu-1} T^{\nu}\right| &= \left|(1 - T)^{-1}(1 - T)\frac{1}{\mu}\sum_{\nu=0}^{\mu-1} T^{\nu}\right| \\ &= \left|(1 - T)^{-1}\frac{1}{\mu}(1 - T^{\mu})\right| \leqq \left|(1 - T)^{-1}\right|\frac{2}{\mu}. \end{aligned}$$

Therefore we conclude by Theorem 26.6

$$\mathrm{D}\text{-}\lim_{\mu\to\infty} [1 - T]\frac{1}{\mu}\sum_{\nu=0}^{\mu-1} T^{\nu} = 0.$$

Since

$$(1 - [1 - T])\frac{1}{\mu}\sum_{\nu=0}^{\mu-1} T^{\nu} = 1 - [1 - T],$$

we obtain therefore

$$\mathrm{D}\text{-}\lim_{\mu\to\infty} \frac{1}{\mu}\sum_{\nu=0}^{\mu-1} T^{\nu} = 1 - [1 - T].$$

On the other hand, recalling the formula §25(31), we obtain

$$(1 - [1 - T])\mathfrak{H} = \{x : Tx = x\}.$$

Thus we have by Theorem 11.6

$$1 - [1 - T] = P_{\{x:\ Tx = x\}}.$$

Therefore we can state

Theorem 51.1. *For a bounded dilatator T by $\mathfrak{R}$, if $\|T\| \leqq 1$, then we have*

$$\mathrm{D}\text{-}\lim_{\mu\to\infty} \frac{1}{\mu}\sum_{\nu=0}^{\mu-1} T^{\nu} = P_{\{x\,:\,Tx = x\}}.$$

Recalling Theorem 26.8, we conclude from this Theorem 51.1

Theorem 51.2. *For a bounded dilatator T by $\mathfrak{R}$, if $\|T\| \leqq 1$,*

then we have for every $x \in R$

$$(\mathcal{R})\text{-}\lim_{\mu\to\infty} \frac{1}{\mu}\sum_{\nu=0}^{\mu-1} T^{\nu}x = P_{\{x\,:\,Tx=x\}}x .$$

Let N be a bounded normal operator on R. N is obviously closed, and furthermore regular by Theorem 32.3. Therefore there exists by Theorem 34.1 a closed ring of projection operators containing 1, by which N is a dilatator. Since the ring convergence implies the convergence by Theorem 21.6, we conclude hence by Theorem 51.2

Theorem 51.3. For a bounded normal operator N on R, if we have $\|N\| \leqq 1$, then we obtain for every $x \in R$

$$\lim_{\mu\to\infty} \frac{1}{\mu}\sum_{\nu=0}^{\mu-1} N^{\nu}x = P_{\{x\,:\,Nx=x\}}x .$$

§52 Translators

Let $\mathcal{R}$ be a ring of projection operators containing 1 in a Hilbert space R. A unitary operator U on R is said to be a translator of $\mathcal{R}$, if

$$U\,\mathcal{R}\,U^{*} = \mathcal{R} .$$

If $\mathcal{R}$ is not closed, then we see easily by the formulas (2) and (3) in §41 that every translator of $\mathcal{R}$ also is a translator of the closed ring $\overline{\mathcal{R}}$ generated by $\mathcal{R}$. Thus we consider the case where $\mathcal{R}$ is closed in the sequel.

Let U be a translator of a closed ring $\mathcal{R}$. For each $x \in R$, we have obviously by the formula §41(1)

$$[Ux] = U[x]U^{*}$$

and hence

$$C_{[Ux]} = U\,C_{[x]}\,U^{*} .$$

An element $a \in R$ is said to be an invariant element of U, if we have $Ua = a$. For every invariant element a of U we have obviously

$$U[a] = [a]U , \qquad U\,C_{[a]} = C_{[a]}\,U .$$

For two symmetric dilatators H_1 and H_2 by $\mathcal{R}$ we define $H_1 \vee H_2$ and $H_1 \wedge H_2$ to mean

$$H_1 \vee H_2 = \int \mathrm{Max}\{(H_1,\xi),(H_2,\xi)\}\,d\xi ,$$

$$H_1 \cap H_2 = \int \mathrm{Min}\{(H_1, \mathfrak{x}), (H_2, \mathfrak{x})\}\, d\mathfrak{x}.$$

For a symmetric dilatator H by $\mathfrak{R}$ we define H^+ and H^- to mean

$$H^+ = H \cup 0, \qquad H^- = (-H)^+.$$

With this definition we see easily by the formula §25(36) that we have

$$[H^+][H^-] = 0, \qquad [H] = [H^+] + [H^-],$$
$$H[H^+] = H^+, \qquad H[H^-] = -H^-.$$

Let H be a symmetric dilatator by $\mathfrak{R}$ in the sequel and we also make use of the notation:

$$H_{(\nu)} = \frac{1}{\nu} \sum_{\rho=0}^{\nu-1} \oplus\, U^\rho H U^{a\rho} \qquad (\nu = 1, 2, \ldots).$$

With this notation we have obviously

$$(M) \qquad \nu H_{(\nu)} = \mu H_{(\mu)} \oplus (\nu - \mu) U^\mu H_{(\nu-\mu)} U^{a\mu} \quad \text{for } \nu \geqq \mu.$$

Theorem 52.1. *For an invariant element a of U we have*

$$(Ha, a) \geqq \Big(\bigwedge_{\rho=1}^{\nu} [H_{(\rho)}^-] Ha, a\Big),$$

if Ha has any sense.

Proof. Putting $P_0 = 1 - [H_{(1)}^-]$,

$$P_\nu = (1 - [H_{(\nu+1)}^-]) \bigwedge_{\rho=1}^{\nu} [H_{(\rho)}^-] \qquad (\nu = 1, 2, \ldots),$$

we have obviously

$$\sum_{\rho=0}^{\nu} P_\rho = 1 - \bigwedge_{\rho=1}^{\nu+1} [H_{(\rho)}^-],$$
$$P_\nu \leqq [H_{(\mu)}^-] \qquad \text{for } \nu \geqq \mu,$$
$$P_\nu [H_{(\nu+1)}^-] = 0.$$

For $\nu > \mu$, we have by (M)

$$\nu H_{(\nu)} P_{\nu-1} = \mu H_{(\mu)} P_{\nu-1} \oplus (\nu - \mu) U^\mu H_{(\nu-\mu)} U^{a\mu} P_{\nu-1}.$$

Since

$$H_{(\nu)} P_{\nu-1} = H_{(\nu)} [H_{(\nu)}] P_{\nu-1} = H_{(\nu)} [H_{(\nu)}^+] P_{\nu-1}$$
$$= H_{(\nu)}^+ P_{\nu-1} \geqq 0,$$
$$H_{(\mu)} P_{\nu-1} = H_{(\mu)} [H_{(\mu)}^-] P_{\nu-1} = -H_{(\mu)}^- P_{\nu-1} \leqq 0,$$

we conclude easily

$$U^\mu H_{(\nu-\mu)} U^{a\mu} P_{\nu-1} \geqq 0$$

and hence

$$H_{(\nu-\mu)}(U^{a\mu} P_{\nu-1} U^\mu) = U^{a\mu}(U^\mu H_{(\nu-\mu)} U^{a\mu} P_{\nu-1}) U^\mu \geqq 0.$$

Putting $\rho = \nu - \mu$, we obtain therefore

$$[H_{(\rho)}^-](U^{a\nu-\rho} P_{\nu-1} U^{\nu-\rho}) = 0 \qquad \text{for } \nu \geqq \rho \geqq 1.$$

This relation yields

$$P_\mu (U^{*\nu-\rho} P_{\nu-1} U^{\nu-\rho}) = 0 \qquad \text{for } \nu, \mu \geqq \rho \geqq 1,$$

because $P_\mu \leqq [H_{(\rho)}^-]$ for $\mu \geqq \rho$. For an arbitrary $\varkappa = 1, 2, \ldots,$ putting $Q_\varkappa = P_\varkappa$,

$$Q_\mu = P_\mu \Big(1 - \bigcup_{\substack{\mu < \nu \leqq \varkappa \\ \rho \leqq \nu}} U^{*\nu-\rho} Q_\nu U^{\nu-\rho}\Big) \qquad (\mu = 0, 1, \ldots, \varkappa - 1),$$

we have obviously $Q_\mu \leqq P_\mu$ and hence

$$U^{*\mu-\rho} Q_\mu U^{\mu-\rho} \leqq U^{*\mu-\rho} P_\mu U^{\mu-\rho} \quad \text{for } \mu \geqq \rho.$$

Therefore we obtain

$$Q_\mu (U^{*\nu-\lambda} Q_\nu U^{\nu-\lambda}) = 0 \qquad \text{for } \mu > \lambda,\ \nu \geqq \lambda.$$

Furthermore it is obvious that

$$Q_\mu (U^{*\nu-\lambda} Q_\nu U^{\nu-\lambda}) = 0 \qquad \text{for } \nu > \mu,\ \nu \geqq \lambda.$$

Therefore we see easily that

$$(U^{*\nu-\lambda} Q_\nu U^{\nu-\lambda})\,(U^{*\mu-\rho} Q_\mu U^{\mu-\rho}) = 0$$

if $\nu \neq \mu$ or $\lambda \neq \rho$.

Since $[H_{(\rho)}^-]\,(U^{*\nu-\rho} P_{\nu-1} U^{\nu-\rho}) = 0$ for $\nu \geqq \rho \geqq 1$, we have

$$U^{*\mu-\rho} P_\mu U^{\mu-\rho} \leqq 1 - [H_{(\rho+1)}^-] \leqq 1 - \bigwedge_{\nu=1}^{\rho+1} [H_{(\nu)}^-] = \sum_{\nu=0}^{\rho} P_\nu,$$

and hence

$$U^{*\mu-\rho} Q_\mu U^{\mu-\rho} \leqq \sum_{\nu=0}^{\mu} P_\nu \qquad \text{for } 0 \leqq \rho \leqq \mu \leqq \varkappa.$$

On the other hand we have for $0 \leqq \mu < \varkappa$

$$P_\mu = Q_\mu + P_\mu \bigcup_{\substack{\mu < \nu \leqq \varkappa \\ \rho < \nu}} U^{*\nu-\rho} Q_\nu U^{\nu-\rho}$$

$$\leqq \sum_{0 \leqq \rho \leqq \mu \leqq \varkappa} U^{*\mu-\rho} Q_\mu U^{\mu-\rho}.$$

Thus we obtain

$$\sum_{0 \leqq \rho \leqq \mu \leqq \varkappa} U^{*\mu-\rho} Q_\mu U^{\mu-\rho} = \sum_{\nu=0}^{\mu} P_\nu = 1 - \bigwedge_{\nu=1}^{\varkappa+1} [H_{(\nu)}^-].$$

Since $U a = a$, we have therefore

$$\begin{aligned}((1 - \bigwedge_{\nu=1}^{\varkappa+1} [H_{(\nu)}^-])\, Ha, a) &= \sum_{\mu=0}^{\varkappa} \sum_{\rho=0}^{\mu} (U^{*\mu-\rho} Q_\mu U^{\mu-\rho} Ha, a) \\ &= \sum_{\mu=0}^{\varkappa} \sum_{\rho=0}^{\mu} (U^{\mu-\rho} H U^{*\mu-\rho} Q_\mu a, a) \\ &= \sum_{\mu=0}^{\varkappa} (\mu+1)\,(H_{(\mu+1)} Q_\mu a, a).\end{aligned}$$

Since $[H_{(\mu+1)}^-] P_\mu = 0$ and $Q_\mu \leqq P_\mu$, we have

$$H_{(\mu+1)} Q_\mu = H_{(\mu+1)} [H_{(\mu+1)}^+] Q_\mu = H_{(\mu+1)}^+ Q_\mu,$$

and hence

$$(H_{(\mu+1)}\, Q_\mu a, a) = (H_{(\mu+1)}^{+}\, Q_\mu a, a) \geqq 0,$$

because both $H_{(\mu+1)}^{+}$ and Q_μ are positive dilatators by $\mathfrak{R}$. Thus we obtain

$$(Ha, a) \geqq ((\bigcap_{\nu=1}^{\mu+1} [H_{(\nu)}^{-}])Ha, a).$$

For a sequence of symmetric dilatators T_ν $(\nu = 1, 2, \ldots)$ by $\mathfrak{R}$, if there is a symmetric dilatator T_0 by $\mathfrak{R}$ for which $T_\nu \leqq T_0$ for every $\nu = 1, 2, \ldots$, then we see easily by Theorem 26.1 that

$$\sup_{\nu \geqq 1} (T_\nu, \mathfrak{F})$$

is finite continuous in some open dense set of the proper space $\mathfrak{F}_{\mathfrak{R}}$ of $\mathfrak{R}$. Thus, putting

$$T = \int \sup_{\nu \geqq 1} (T_\nu, \mathfrak{F})\, d\mathfrak{F},$$

we obtain a symmetric dilatator T by $\mathfrak{R}$. This symmetric dilatator T will be denoted by

$$\bigcup_{\nu=1}^{\infty} T_\nu.$$

With this definition we have

Theorem 52.2. *If a symmetric dilatator T is commutative to U and both Ha and Ta have any sense for an invariant element a of U, then $\bigcup_{\nu=1}^{\infty} (H_{(\nu)} \wedge T) = T$ implies $(Ha, a) \geqq (Ta, a)$.*

Proof. For an arbitrary $\varepsilon > 0$, putting

$$K = H - T + \varepsilon,$$

we obtain a symmetric dilatator K by $\mathfrak{R}$ and we have

$$K_{(\nu)} = H_{(\nu)} - T + \varepsilon \qquad (\nu = 1, 2, \ldots),$$

becaue T is commutative to U by assumption. By virtue of Theorem 52.1, we have then for every $\varkappa = 1, 2, \ldots$

$$(Ka, a) \geqq ((\bigcap_{\nu=1}^{\varkappa} [K_{(\nu)}^{-}])Ka, a).$$

Thus, putting $P = \bigcap_{\nu=1}^{\infty} [K_{(\nu)}^{-}]$, we obtain by Theorem 13.5

$$(Ka, a) \geqq (PKa, a).$$

We shall now prove that $P = 0$. Since $P \leqq [K_{(\nu)}^{-}]$ for every $\nu = 1, 2, \ldots$, we have

$$K_{(\nu)} P = K_{(\nu)} [K_{(\nu)}^{-}] P = -K_{(\nu)}^{-} P \leqq 0,$$

and hence

$$TP - \varepsilon P \geqq H_{(\nu)} P \geqq (H_{(\nu)} \wedge T) P$$

for every $\nu = 1, 2, \ldots$. Thus we obtain

$$T P - \varepsilon P \geqq \left(\bigcup_{\nu=1}^{\infty} (H_{(\nu)} \wedge T) \right) P = T P .$$

This relation yields $P = 0$, because $\varepsilon > 0$. Therefore we have $(K a, a) \geqq 0$, and hence

$$(H a, a) \geqq (T a, a) - \varepsilon \| a \|^2 .$$

Since $\varepsilon > 0$ may be arbitrary, we obtain hence our assertion.

For a sequence of symmetric dilatators T_ν $(\nu = 1, 2, \ldots)$, if there is a symmetric dilatator T_0 by $\mathfrak{K}$ such that

$$| T_\nu | \leqq T_0 \qquad \text{for every } \nu = 1, 2, \ldots,$$

then we see easily by Theorem 26.1 that both

$$\overline{\lim_{\nu \to \infty}} (T_\nu, \mathfrak{f}) \quad \text{and} \quad \underline{\lim_{\nu \to \infty}} (T_\nu, \mathfrak{f})$$

are finite continuous in some open dense set of $\mathfrak{C}_{\mathfrak{K}}$. Thus we define $\overline{\lim_{\nu \to \infty}} T_\nu$ and $\underline{\lim_{\nu \to \infty}} T_\nu$ to mean

$$\overline{\lim_{\nu \to \infty}} T_\nu = \int \overline{\lim_{\nu \to \infty}} (T_\nu, \mathfrak{f}) \, d\mathfrak{f} ,$$

$$\underline{\lim_{\nu \to \infty}} T_\nu = \int \underline{\lim_{\nu \to \infty}} (T_\nu, \mathfrak{f}) \, d\mathfrak{f} .$$

With this definition we see at once that T_ν $(\nu = 1, 2, \ldots)$ is D-convergent if and only if

$$\overline{\lim_{\nu \to \infty}} T_\nu = \underline{\lim_{\nu \to \infty}} T_\nu .$$

For an arbitrary $\mu = 1, 2, \ldots$, putting

$$\overline{H} = \overline{\lim_{\nu \to \infty}} ((H_{(\nu)} \wedge \mu) \vee (-\mu)),$$

$$\underline{H} = \underline{\lim_{\nu \to \infty}} ((H_{(\nu)} \wedge \mu) \vee (-\mu)),$$

we obtain two dilatators $\overline{H}$ and $\underline{H}$ by $\mathfrak{K}$. Both $\overline{H}$ and $\underline{H}$ are commutative to U. In fact we have by definition

$$\begin{aligned} U \overline{H} U^* &= \overline{\lim_{\nu \to \infty}} ((U H_{(\nu)} U^* \wedge \mu) \vee (-\mu)) \\ &= \overline{\lim_{\nu \to \infty}} \left\{ \frac{\nu+1}{\nu} \left((H_{(\nu+1)} \wedge \frac{\nu}{\nu+1} \mu) \vee (-\frac{\nu}{\nu+1} \mu) \right) - \frac{1}{\nu} H \right\} \\ &= \overline{\lim_{\nu \to \infty}} ((H_{(\nu+1)} \wedge \mu) \vee (-\mu)) = \overline{H} , \end{aligned}$$

and we obtain likewise $U \underline{H} U^* = \underline{H}$.

Since $\underline{H} \leqq \overline{H}$ and

$$\left(\bigcup_{\nu=1}^{\infty} (H_{(\nu)} \wedge \overline{H}) \right)$$

$$= ((\bigvee_{\nu=1}^{\infty} (H_{(\nu)} \wedge \mu) \vee (-\mu)) \wedge \bar{H}) \vee \underline{H} \geq \bar{H},$$

we see easily by definition that

$$[\bar{H} - \underline{H}]\bar{H} \leq [\bar{H} - \underline{H}] \bigvee_{\nu=1}^{\infty} (H_{(\nu)} \wedge \bar{H})$$
$$= \bigvee_{\nu=1}^{\infty} (([\bar{H} - \underline{H}] H)_{(\nu)} \wedge [\bar{H} - \underline{H}]\bar{H}),$$

because

$$U[\bar{H} - \underline{H}]U^* = [U(\bar{H} - \underline{H})U^*] = [\bar{H} - \underline{H}].$$

Thus we obtain by Theorem 52.2

$$([\bar{H} - \underline{H}]Ha, a) \geq ([\bar{H} - \underline{H}]\bar{H}a, a).$$

Since

$$\underline{H} = -\overline{\lim_{\nu \to \infty}} (((-H)_{(\nu)} \wedge \mu) \vee (-\mu)),$$

we obtain likewise

$$([\bar{H} - \underline{H}]Ha, a) \leq ([\bar{H} - \underline{H}]\underline{H}a, a).$$

Since $\underline{H} \leq \bar{H}$, we conclude therefore

$$([\bar{H} - \underline{H}]\underline{H}a, a) = ([\bar{H} - \underline{H}]\bar{H}a, a),$$

and hence $((\bar{H} - \underline{H})a, a) = 0$. Since $\bar{H} - \underline{H} \geq 0$, we obtain hence by Theorem 33.4 $(\bar{H} - \underline{H})a = 0$, and hence by Theorem 25.1

$$\bar{H} C_{[a]} = \underline{H} C_{[a]}.$$

Furthermore we have

$$\bar{H} C_{[a]} = \overline{\lim_{\nu \to \infty}} (((H C_{[a]})_{(\nu)} \wedge \mu) \vee (-\mu)),$$

$$\underline{H} C_{[a]} = \underline{\lim_{\nu \to \infty}} (((H C_{[a]})_{(\nu)} \wedge \mu) \vee (-\mu)),$$

because $U C_{[a]} U^* = C_{[a]}$. Therefore we conclude that the sequence

$$((H C_{[a]})_{(\nu)} \wedge \mu) \vee (-\mu) \qquad (\nu = 1, 2, \ldots)$$

is D-convergent for every $\mu = 1, 2, \ldots$.

On the other hand we have

$$(H C_{[a]})_{(\nu)} a = \frac{1}{\nu} \sum_{\mu=0}^{\nu-1} U^{\mu} H C_{[a]} U^{*\mu} a = \frac{1}{\nu} \sum_{\mu=0}^{\nu-1} U^{\mu} Ha.$$

Thus we can find by Theorem 50.1 an element $f \in R$ for which

$$\lim_{\nu \to \infty} (H C_{[a]})_{(\nu)} a = f,$$

and hence there exists by Theorem 21.6 a subsequence ν_{μ} $(\mu = 1, 2, \ldots)$ for which

$$(\mathfrak{R})\text{-}\lim_{\mu \to \infty} (H C_{[a]})_{(\nu_{\mu})} a = f.$$

Thus we have by Theorem 21.8

$$\lim_{\mu \to \infty} ((H C_{[a]})_{(\nu_{\mu})}, g) = (\frac{f}{a}, g)$$

in some open set being dense in $\mathfrak{U}\mathfrak{C}_{[a]}$. Therefore we obtain

$$\mathrm{D}\text{-}\lim_{\nu\to\infty}(H\mathfrak{C}_{[a]})_{(\nu)} = \int_{\mathfrak{C}_{[a]}} \left(\frac{\xi}{a},\mathfrak{x}\right) d\mathfrak{x}.$$

Let T be a dilatator by $\mathfrak{R}$ such that Ta has a sense. Putting $H = T \oplus T^*$, $K = (-i)T \oplus iT^*$, we obtain symmetric dilatators H and K by $\mathfrak{R}$ for which both Ha and Ka have sense, and

$$T = H \oplus iK.$$

Therefore we can state

Theorem 52.3. *For an invariant element a of a translator $\mathfrak{U}$ of $\mathfrak{R}$ and a dilatator T by $\mathfrak{R}$, the sequence*

$$\frac{1}{\nu}\sum_{\mu=0}^{\nu-1}\mathfrak{U}^{\mu}\,T\,\mathfrak{U}^{*\mu}\,\mathfrak{C}_{[a]} \qquad (\nu = 1, 2, \ldots)$$

is D-convergent, if Ta has any sense.

For every $x \in R$, putting

$$T = \int_{\mathfrak{C}_{[a]}} \left(\frac{x}{a},\mathfrak{x}\right) d\mathfrak{x},$$

we obtain a dilatator T by $\mathfrak{R}$ and Ta has sense. Since

$$\mathfrak{U}^{\mu}\,T\,\mathfrak{U}^{*\mu}\mathfrak{C}_{[a]}a = \mathfrak{U}^{\mu}Ta = \mathfrak{U}^{\mu}[a]x,$$

we obtain therefore by Theorems 266.6 and 51.3

Theorem 52.4. *For an invariant element a of a translator $\mathfrak{U}$ of $\mathfrak{R}$ we can find a sequence $\mathfrak{R} \ni P_\rho \uparrow_{\rho=1}^{\infty} \mathfrak{C}_{[a]}$ such that*

$$(\mathfrak{R})\text{-}\lim_{\nu\to\infty} P_\rho\left(\frac{1}{\nu}\sum_{\mu=0}^{\nu-1}\mathfrak{U}^{\mu}[a]x\right) = P_\rho\, P_{\{x:\,\mathfrak{U}x=x\}}[a]x$$

for every $x \in R$ and $\rho = 1, 2, \ldots$.

§53 Ideal Spaces

Let $\mathfrak{R}$ be a closed ring of projection operators in a Hilbert space R, $\mathfrak{C}_{\mathfrak{R}}$ the proper space of $\mathfrak{R}$, and $\overline{\mathfrak{R}}$ the commutor of $\mathfrak{R}$. For an element $e \in R$ and a point $\mathfrak{x} \in \mathfrak{U}\mathfrak{C}_{[e]}$, we shall denote by $R_{e,\mathfrak{x}}$ the totality of those elements $x \in R$ for which

$$\lim_{P\to\mathfrak{x}} \frac{\|Px\|}{\|Pe\|} < +\infty.$$

We see easily that $R_{e,\mathfrak{x}}$ is a linear manifold of R. For each $a \in R_{e,\mathfrak{x}}$ we shall denote by $a^{e,\mathfrak{x}}$ the totality of those elements $x \in R_{e,\mathfrak{x}}$ for which

$$\lim_{P \to \mathfrak{F}} \frac{\| P(a - x) \|}{\| P e \|} = 0.$$

Then we have obviously by definition that

$$a^{e,\mathfrak{F}} = l^{e,\mathfrak{F}} \quad \text{if and only if} \quad a \in l^{e,\mathfrak{F}}.$$

Furthermore we see easily that $x \in a^{e,\mathfrak{F}}$, $y \in l^{e,\mathfrak{F}}$ implies

$$\alpha x + \beta y \in (\alpha a + \beta l)^{e,\mathfrak{F}}$$

for every complex numbers α, β. Thus we define $\alpha a^{e,\mathfrak{F}} + \beta l^{e,\mathfrak{F}}$ to mean

$$\alpha a^{e,\mathfrak{F}} + \beta l^{e,\mathfrak{F}} = (\alpha a + \beta l)^{e,\mathfrak{F}}.$$

With this definition we obtain a linear space $x^{e,\mathfrak{F}}$ $(x \in R_{e,\mathfrak{F}})$. This linear space will be denoted by $R^{e,\mathfrak{F}}$.

For every $x \in a^{e,\mathfrak{F}}$ and $y \in l^{e,\mathfrak{F}}$ we have

$$\lim_{P \to \mathfrak{F}} \frac{(Px, y)}{\| Pe \|^2} = \lim_{P \to \mathfrak{F}} \frac{(Pa, l)}{\| Pe \|^2},$$

because

$$|(Px, y) - (Pa, l)| \leqq |(P(x - a), Py)| + |(Pa, P(y - l))|$$
$$\leqq \| P(x - a) \| \| Py \| + \| Pa \| \| P(y - l) \|.$$

Therefore we define $(a^{e,\mathfrak{F}}, l^{e,\mathfrak{F}})$ to mean

$$(1) \qquad (a^{e,\mathfrak{F}}, l^{e,\mathfrak{F}}) = \lim_{P \to \mathfrak{F}} \frac{(Pa, l)}{\| Pe \|^2}.$$

With this definition we see easily that $(a^{e,\mathfrak{F}}, l^{e,\mathfrak{F}})$ is an inner product on $R^{e,\mathfrak{F}}$, and hence $R^{e,\mathfrak{F}}$ is an inner product space by this inner product. This inner product space $R^{e,\mathfrak{F}}$ is called the <u>ideal space</u> of R at $\mathfrak{F}$ by e.

Concerning the ideal space $R^{e,\mathfrak{F}}$, we have obviously by (1)

$$(2) \qquad \| x^{e,\mathfrak{F}} \| = \lim_{P \to \mathfrak{F}} \frac{\| Px \|}{\| Pe \|}.$$

Consequently we have

$$(3) \qquad \| e^{e,\mathfrak{F}} \| = 1.$$

For each $x \in R_{e,\mathfrak{F}}$, we also have $Qx \in R_{e,\mathfrak{F}}$ for every $Q \in \overline{\mathcal{R}}$, because we have for every $P \in \mathcal{R}$

$$\| PQx \| = \| QPx \| \leqq \| Px \|.$$

Furthermore we have obviously by definition

$$(4) \qquad (Px)^{e,\mathfrak{F}} = x^{e,\mathfrak{F}} \qquad \text{for } \mathfrak{F}^* \in \mathfrak{V}_P, \ P \in \mathcal{R}.$$

If $x^{e,\mathfrak{F}} \neq 0$, then for any $y \in R_{e,\mathfrak{F}}$ we have by §19(12)

$$|(\frac{y}{x}, \mathfrak{F})| = \lim_{P \to \mathfrak{F}} \frac{\| P[x]y \|}{\| Px \|}$$

$$= \lim_{P \to \mathfrak{F}} \frac{\| Pe \|}{\| Px \|} \lim_{P \to \mathfrak{F}} \frac{\| P[x]y \|}{\| Pe \|} < +\infty .$$

Thus, putting $\alpha = (\frac{y}{x}, \mathfrak{F})$, we obtain

$$\lim_{P \to \mathfrak{F}} \frac{\| P([x]y - \alpha x) \|}{\| Pe \|}$$

$$= \lim_{P \to \mathfrak{F}} \frac{\| Px \|}{\| Pe \|} \lim_{P \to \mathfrak{F}} \frac{\| P([x]y - \alpha x) \|}{\| Px \|} = 0,$$

because we have by the formulas (4), (5), and (12) in §19

$$\lim_{P \to \mathfrak{F}} \frac{\| P([x]y - \alpha x) \|}{\| Px \|}$$

$$= |(\frac{[x]y - \alpha x}{x}, \mathfrak{F})| = |(\frac{y}{x}, \mathfrak{F}) - \alpha| = 0.$$

Consequently we obtain $([x]y)^{e,\mathfrak{F}} = \alpha x^{e,\mathfrak{F}}$ by definition. Thus we have

$$(5) \qquad ([x]y)^{e,\mathfrak{F}} = (\frac{y}{x}, \mathfrak{F}) x^{e,\mathfrak{F}} \qquad \text{for } x, y \in R_{e,\mathfrak{F}},$$

adopting the convention $\infty \cdot 0 = 0$.

If $(x^{e,\mathfrak{F}}, y^{e,\mathfrak{F}}) = 0$, then we have by (5)

$$\| ([x]y)^{e,\mathfrak{F}} \|^2 = \lim_{P \to \mathfrak{F}} \frac{(P[x]y, [x]y)}{\| Pe \|^2}$$

$$= \lim_{P \to \mathfrak{F}} \frac{(P[x]y, y)}{\| Pe \|^2} = ((x]y)^{e,\mathfrak{F}}, y^{e,\mathfrak{F}})$$

$$= (\frac{y}{x}, \mathfrak{F}) (x^{e,\mathfrak{F}}, y^{e,\mathfrak{F}}) = 0.$$

Conversely if $([x]y)^{e,\mathfrak{F}} = 0$, then we have by §14(3)

$$(x^{e,\mathfrak{F}}, y^{e,\mathfrak{F}}) = \lim_{P \to \mathfrak{F}} \frac{(Px, y)}{\| Pe \|^2} = \lim_{P \to \mathfrak{F}} \frac{(P[x]x, y)}{\| Pe \|^2}$$

$$= \lim_{P \to \mathfrak{F}} \frac{(Px, [x]y)}{\| Pe \|^2} = (x^{e,\mathfrak{F}}, ([x]y)^{e,\mathfrak{F}}) = 0.$$

Therefore we have

$$(6) \qquad (x^{e,\mathfrak{F}}, y^{e,\mathfrak{F}}) = 0 \text{ if and only if } ([x]y)^{e,\mathfrak{F}} = 0.$$

Theorem 53.1. If $x^{e,\mathfrak{F}} \neq 0$, then we can find $y \in R_{e,\mathfrak{F}}$ such that $x^{e,\mathfrak{F}} = y^{e,\mathfrak{F}}$, $[y] = C_{[e]}[x]$, and

$$\| Py \| = \| x^{e,\mathfrak{F}} \| \| P C_{[x]} e \| \qquad \text{for every } P \in \mathfrak{K}.$$

Proof. If $x^{e,\mathfrak{F}} \neq 0$, then we have obviously $C_{[x]} e \in \mathfrak{F}$ by definition, and hence by the formula §14(16')

$$C_{[C_{[x]}e]} = C_{[C_{[e]}x]} = C_{[e]} C_{[x]} \in \mathfrak{F}.$$

Thus $[C_{[x]}e]\mathfrak{R}$ is isomorphic to $[C_{[e]}x]\mathfrak{R}$. Furthermore both $[C_{[x]}e]\mathfrak{R}$ and $[C_{[e]}x]\mathfrak{R}$ are simple by Theorem 20.5. Therefore we can find by Theorem 41.4 an element $a \in [C_{[e]}x]\mathfrak{R}$ such that

$$\| P a \| = \| P C_{[x]} e \| \qquad \text{for every } P \in \mathfrak{R}.$$

For such a we have

$$[a] = [C_{[e]}x].$$

Because $Pa = 0$, $P \in \mathfrak{R}$ implies $P C_{[x]} e = 0$, and hence by the formula §14(19) $P C_{[e]} x = 0$. Thus we have

$$C_{[a]} \geq C_{[C_{[e]}x]}.$$

On the other hand we have obviously $[a] \leq [C_{[e]}x]$, and hence we conclude by Theorem 19.3 $[a] = [C_{[e]}x]$.

Since $C_{[x]} \in \mathfrak{F}$, we obtain further by the formulas (2), (3), and (4)

$$\| a^{e,\mathfrak{F}} \| = \| (C_{[x]} e)^{e,\mathfrak{F}} \| = \| e^{e,\mathfrak{F}} \| = 1.$$

Since $[a] = C_{[e]}[x] \leq [x]$ by the formula §14(6'), we obtain by Theorem 19.4

$$[a]x = C_{[a]}[x]x = C_{[a]}x,$$

and hence $([a]x)^{e,\mathfrak{F}} = x^{e,\mathfrak{F}}$ by (4). Thus we obtain by (5)

$$x^{e,\mathfrak{F}} = (\tfrac{x}{a}, \mathfrak{F})\, a^{e,\mathfrak{F}}.$$

This relation yields

$$\| x^{e,\mathfrak{F}} \| = |(\tfrac{x}{a}, \mathfrak{F})| \, \| a^{e,\mathfrak{F}} \| = |(\tfrac{x}{a}, \mathfrak{F})|.$$

Therefore, putting

$$y = (\tfrac{x}{a}, \mathfrak{F})\, a,$$

we have obviously $x^{e,\mathfrak{F}} = y^{e,\mathfrak{F}}$, and $[y] = [a]$, because we have $x^{e,\mathfrak{F}} \neq 0$ by assumption. Furthermore we have

$$\| P y \| = |(\tfrac{x}{a}, \mathfrak{F})| \, \| P a \| = \| x^{e,\mathfrak{F}} \| \, \| P C_{[x]} e \|$$

for every $P \in \mathfrak{R}$.

Theorem 53.2. If $(Qx)^{e,\mathfrak{F}} = 0$, $Q \in \overline{\mathfrak{R}}$, then we can find an element $y \in R_{e,\mathfrak{F}}$ such that

$$x^{e,\mathfrak{F}} = y^{e,\mathfrak{F}}, \quad Q[y] = 0, \quad y \prec \| x^{e,\mathfrak{F}} \| e.$$

Proof. We need only consider the case where $x^{e,\mathfrak{F}} \neq 0$. Putting $a = (1 - Q)x$, we have $Q[a] = 0$ by the formula §14(1), and further by assumption

$$a^{e,\mathfrak{F}} = x^{e,\mathfrak{F}} - (Qx)^{e,\mathfrak{F}} = x^{e,\mathfrak{F}}.$$

Since $x^{e,\mathfrak{F}} \neq 0$ by assumption, we can find by Theorem 53.1 an element $y \in R_{e,\mathfrak{F}}$ such that $a^{e,\mathfrak{F}} = y^{e,\mathfrak{F}}$, $[y] = C_{[e]}[a]$ and

$$\|Py\| = \|a^{e,\mathfrak{F}}\| \|PC_{[a]}e\| \quad \text{for every } P \in \mathfrak{R}.$$

For such y, we have obviously $x^{e,\mathfrak{F}} = y^{e,\mathfrak{F}}$, $Q[y] = 0$ and

$$y \prec \|x^{e,\mathfrak{F}}\| C_{[a]}e \prec \|x^{e,\mathfrak{F}}\| e.$$

Theorem 53.3. *For a sequence* $x_\nu \in R_{e,\mathfrak{F}}$ $(\nu = 1, 2, \ldots)$, *if for any* $\varepsilon > 0$ *we can find* ν_0 *such that*

$$x_\nu - x_\mu \prec \varepsilon e \qquad \text{for } \nu, \mu \geqq \nu_0,$$

then $x_\nu^{e,\mathfrak{F}}$ $(\nu = 1, 2, \ldots)$ *is convergent in the ideal space* $R^{e,\mathfrak{F}}$ *and for* $x = \lim_{\nu\to\infty} x_\nu$, *we have*

$$x^{e,\mathfrak{F}} = \lim_{\nu\to\infty} x_\nu^{e,\mathfrak{F}}.$$

Proof. From assumption we conclude easily that for any $\varepsilon > 0$ we can find ν_0 such that

$$x - x_\nu \prec \varepsilon e \qquad \text{for } \nu \geqq \nu_0.$$

Thus we have by definition $x - x_\nu \in R_{e,\mathfrak{F}}$ and

$$\|(x - x_\nu)^{e,\mathfrak{F}}\| \leqq \varepsilon \qquad \text{for } \nu \geqq \nu_0.$$

Since $x_\nu \in R_{e,\mathfrak{F}}$ for every $\nu = 1, 2, \ldots$, we conclude hence that we have $x \in R_{e,\mathfrak{F}}$ and

$$\|x^{e,\mathfrak{F}} - x_\nu^{e,\mathfrak{F}}\| \leqq \varepsilon \qquad \text{for } \nu \geqq \nu_0.$$

Therefore we have $x^{e,\mathfrak{F}} = \lim_{\nu\to\infty} x_\nu^{e,\mathfrak{F}}$.

Theorem 53.4. *The ideal space* $R^{e,\mathfrak{F}}$ *is a Hilbert space, that is,* $R^{e,\mathfrak{F}}$ *is complete.*

Proof. Let $a_\nu^{e,\mathfrak{F}}$ $(\nu = 1, 2, \ldots)$ be an orthonormal system in $R^{e,\mathfrak{F}}$. We shall prove firstly by induction that there is a sequence $x_\nu \in R_{e,\mathfrak{F}}$ $(\nu = 1, 2, \ldots)$ such that

$$a_\nu^{e,\mathfrak{F}} = x_\nu^{e,\mathfrak{F}}, \qquad x_\nu \prec e \quad (\nu = 1, 2, \ldots),$$
$$[x_\nu][x_\mu] = 0 \qquad \text{for } \nu \neq \mu.$$

Putting $\chi_1 = a_1$, we suppose that there exists χ_ν ($\nu = 1, 2, \ldots, \varkappa$) such that $a_\nu^{e,\mathfrak{F}} = \chi_\nu^{e,\mathfrak{F}}$, $\chi_\nu \prec e$ ($\nu = 1, 2, \ldots, \varkappa$), and

$$[\chi_\nu][\chi_\mu] = 0 \qquad \text{for } \nu \neq \mu .$$

Then we have by assumption

$$(a_{\varkappa+1}^{e,\mathfrak{F}}, \chi_\nu^{e,\mathfrak{F}}) = 0 \qquad \text{for every } \nu = 1, 2, \ldots, \varkappa,$$

and hence by (6)

$$([\chi_\nu] a_{\varkappa+1})^{e,\mathfrak{F}} = 0 \qquad \text{for every } \nu = 1, 2, \ldots, \varkappa.$$

This relation yields

$$\Big(\sum_{\nu=1}^{\varkappa} [\chi_\nu] a_{\varkappa+1}\Big)^{e,\mathfrak{F}} = 0.$$

Since $[\chi_\nu][\chi_\mu] = 0$ for $\nu \neq \mu$ by assumption, we have by Theorem 12.6

$$\sum_{\nu=1}^{\varkappa} [\chi_\nu] = \bigcup_{\nu=1}^{\varkappa} [\chi_\nu] \in \overline{\mathcal{R}}.$$

Thus we can find by Theorem 53.2 an element $\chi_{\varkappa+1} \in \mathcal{R}_{e,\mathfrak{F}}$ such that

$$a_{\varkappa+1}^{e,\mathfrak{F}} = \chi_{\varkappa+1}^{e,\mathfrak{F}}, \quad \Big(\bigcup_{\nu=1}^{\varkappa} [\chi_\nu]\Big)[\chi_{\varkappa+1}] = 0, \quad \chi_{\varkappa+1} \prec e,$$

because $\| a_{\varkappa+1}^{e,\mathfrak{F}} \| = 1$. Therefore we see by induction that there exists $\chi_\nu \in \mathcal{R}_{e,\mathfrak{F}}$ ($\nu = 1, 2, \ldots$) subject to our requirement.

For a sequence of complex numbers α_ν ($\nu = 1, 2, \ldots$), if

$$\sum_{\nu=1}^{\infty} |\alpha_\nu|^2 < +\infty,$$

then we have for every $P \in \mathcal{R}$

$$\| P \sum_{\nu=\mu}^{\varkappa} \alpha_\nu \chi_\nu \|^2 = \sum_{\nu=\mu}^{\varkappa} |\alpha_\nu|^2 \| P \chi_\nu \|^2 \leqq \| Pe \|^2 \sum_{\nu=\mu}^{\varkappa} |\alpha_\nu|^2,$$

because we have for $\nu \neq \mu$

$$(P\chi_\nu, P\chi_\mu) = (P[\chi_\nu]\chi_\nu, P[\chi_\mu]\chi_\mu) = (P\chi_\nu, [\chi_\nu][\chi_\mu]\chi_\mu) = 0.$$

Thus $\sum_{\nu=1}^{\infty} \alpha_\nu \chi_\nu^{e,\mathfrak{F}}$ is convergent by Theorem 53.3. Therefore we conclude by Theorem 7.10 that $\mathcal{R}^{e,\mathfrak{F}}$ is complete.

For each $\chi \in \mathcal{R}$, putting

$$\varphi(\mathfrak{F}) = \lim_{P \to \mathfrak{F}} \frac{\| P\chi \|^2}{\| Pe \|^2} \qquad \text{for } \mathfrak{F} \in \mathfrak{O}_{C_{[e]}},$$

we have by Theorem 18.2

$$\| C_{[e]} \chi \|^2 = \int_{C_{[e]}} \varphi(\mathfrak{F}) \| d\mathfrak{F} e \|^2.$$

Thus we have $x \in R_{e,\mathfrak{F}}$ for every $\mathfrak{F}$ except for at most a nowhere dense set in $\cup C_{[e]}$. Furthermore we have by (1)

$$(C_{[e]}x, y) = \int_{C_{[e]}} (x^{e,\mathfrak{F}}, y^{e,\mathfrak{F}}) \| d\mathfrak{F} e \|^2.$$

Therefore we can state

Theorem 53.5. For each $x \in R$, we have $x \in R_{e,\mathfrak{F}}$ for every $\mathfrak{F}$ except for at most a nowhere dense set in $\cup C_{[e]}$ and

$$\| C_{[e]}x \|^2 = \int_{C_{[e]}} \| x^{e,\mathfrak{F}} \|^2 \| d\mathfrak{F} e \|^2,$$

$$(C_{[e]}x, y) = \int_{C_{[e]}} (x^{e,\mathfrak{F}}, y^{e,\mathfrak{F}}) \| d\mathfrak{F} e \|^2.$$

Theorem 53.6. If $(\mathcal{R})\text{-}\lim_{\nu\to\infty} x_\nu = x$, then we have

$$\lim_{\nu\to\infty} x_\nu^{e,\mathfrak{F}} = x^{e,\mathfrak{F}}$$

for every $\mathfrak{F}$ except for at most a nowhere dense set in $\cup C_{[e]}$.

Proof. If $(\mathcal{R})\text{-}\lim_{\nu\to\infty} x_\nu = x$, then there exists by definition an element $l \in R$ such that for any $\varepsilon > 0$ we can find ν_0 for which

$$x_\nu - x \prec \varepsilon l \qquad \text{for } \nu \geq \nu_0.$$

For such $l \in R$, putting

$$\varphi(\mathfrak{F}) = \lim_{P\to\mathfrak{F}} \frac{\| P l \|^2}{\| P e \|^2} \qquad \text{for } \mathfrak{F} \in \cup C_{[e]},$$

we obtain by Theorem 18.2 a continuous function φ on $\cup C_{[e]}$ such that

$$\| P l \|^2 = \int_P \varphi(\mathfrak{F}) \| d\mathfrak{F} e \|^2 \qquad \text{for } C_{[e]} \geq P \in \mathcal{R}.$$

Since $\{\mathfrak{F} : \varphi(\mathfrak{F}) < \rho\}$ is open, we can find by Theorems 15.6 and 15.9 a sequence $P_\rho \in \mathcal{R}$ $(\rho = 1, 2, \ldots)$ such that

$$\cup P_\rho = \{\mathfrak{F} : \varphi(\mathfrak{F}) < \rho\}^- \qquad (\rho = 1, 2, \ldots).$$

For such P_ρ $(\rho = 1, 2, \ldots)$ we see easily that $P_\rho \uparrow_{\rho=1}^{\infty} C_{[e]}$. Furthermore we have for every $\rho = 1, 2, \ldots$

$$\| P_\rho l \|^2 \leqq \int_{P_\rho} \rho \| d\mathfrak{F} e \|^2 = \rho \| P_\rho e \|^2,$$

and hence

$$P_\rho x_\nu - P_\rho x \prec \varepsilon \rho P_\rho e \text{ for } \nu \geq \nu_0.$$

By virtue of Theorem 53.5, we can find an open set A being dense

in $\mathfrak{U}_{\mathfrak{C}_{[e]}}$ such that $x \in R_{e,\mathfrak{F}}$ for every $\mathfrak{F} \in A$. For such A we have further by (4)

$$(P_\rho x_\nu)^{e,\mathfrak{F}} = x_\nu^{e,\mathfrak{F}} \quad \text{for } \mathfrak{F} \in A\,\mathfrak{U}_{P_\rho},\ \rho = 1, 2, \ldots,$$

and hence we obtain by Theorem 53.3

$$\lim_{\nu\to\infty} x_\nu^{e,\mathfrak{F}} = x^{e,\mathfrak{F}} \quad \text{for } \mathfrak{F} \in \sum_{\rho=1}^{\infty} A\,\mathfrak{U}_{P_\rho}.$$

On the other hand we have by Theorem 15.7

$$\Big(\sum_{\rho=1}^{\infty} A\,\mathfrak{U}_{P_\rho}\Big)^- = \Big(A\Big(\sum_{\rho=1}^{\infty}\mathfrak{U}_{P_\rho}\Big)^-\Big)^-$$
$$= (A\,\mathfrak{U}_{\mathfrak{C}_{[e]}})^- = A^- = \mathfrak{U}_{\mathfrak{C}_{[e]}},$$

§54 Reduction Theory

Let T be a bounded linear operator on a Hilbert space R and commutative to a closed ring of projection operators $\mathfrak{R}$. For every element $x \in R_{e,\mathfrak{F}}$, we have by §53(2)

$$\|(Tx)^{e,\mathfrak{F}}\| = \lim_{P\to\mathfrak{F}} \frac{\|PTx\|}{\|Pe\|} = \lim_{P\to\mathfrak{F}} \frac{\|TPx\|}{\|Pe\|}$$
$$\leqq \|T\|\,\|x^{e,\mathfrak{F}}\|.$$

Thus $x^{e,\mathfrak{F}} = 0$ implies $(Tx)^{e,\mathfrak{F}} = 0$, and hence $x^{e,\mathfrak{F}} = y^{e,\mathfrak{F}}$ implies $(Tx)^{e,\mathfrak{F}} = (Ty)^{e,\mathfrak{F}}$. Therefore, putting

$$(1) \qquad T^{e,\mathfrak{F}}\,x^{e,\mathfrak{F}} = (Tx)^{e,\mathfrak{F}} \qquad \text{for } x \in R_{e,\mathfrak{F}},$$

we obtain an operator $T^{e,\mathfrak{F}}$ on the ideal space $R^{e,\mathfrak{F}}$. This operator $T^{e,\mathfrak{F}}$ is called the <u>reduced operator</u> of T on $R^{e,\mathfrak{F}}$. The reduced operator $T^{e,\mathfrak{F}}$ also is obviously a bounded linear operator on $R^{e,\mathfrak{F}}$ and $\|T^{e,\mathfrak{F}}\| \leqq \|T\|$.

For the adjoint operator T^* of T we have by definition

$$(T^{*e,\mathfrak{F}}\,x^{e,\mathfrak{F}},\ y^{e,\mathfrak{F}}) = ((T^*x)^{e,\mathfrak{F}},\ y^{e,\mathfrak{F}})$$
$$= \lim_{P\to\mathfrak{F}} \frac{(PT^*x,\ y)}{\|Pe\|^2} = \lim_{P\to\mathfrak{F}} \frac{(Px,\ Ty)}{\|Pe\|^2}$$
$$= (x^{e,\mathfrak{F}},\ T^{e,\mathfrak{F}}\,y^{e,\mathfrak{F}})$$

for every $x,\ y \in R_{e,\mathfrak{F}}$. Thus we have

$$(2) \qquad T^{*e,\mathfrak{F}} = T^{e,\mathfrak{F}*}.$$

Therefore we see that if T is symmetric, then the reduced operator $T^{e,\mathfrak{F}}$ also is symmetric.

For two bounded linear operators T_1 and T_2 commutative to $\mathfrak{R}$,

we have obviously by the definition (1)

$$(3) \qquad (T_1 T_2)^{\mathfrak{e},\mathfrak{F}} = T_1^{\mathfrak{e},\mathfrak{F}} T_2^{\mathfrak{e},\mathfrak{F}}.$$

By virtue of the formulas (2) and (3), we see at once that the reduced operators of unitary operators also are unitary operato-

For every $Q \in \overline{\mathfrak{R}}$ we have by (2) and (3)

$$Q^{\mathfrak{e},\mathfrak{F}^*} = Q^{*\mathfrak{e},\mathfrak{F}}, \qquad Q^{\mathfrak{e},\mathfrak{F}} Q^{\mathfrak{e},\mathfrak{F}} = Q^{\mathfrak{e},\mathfrak{F}},$$

and hence the reduced operator $Q^{\mathfrak{e},\mathfrak{F}}$ also is a projection operator on the ideal space $R^{\mathfrak{e},\mathfrak{F}}$.

If $\mathfrak{F} \overline{\in} \cup C_Q$, then $(C_{\mathfrak{R}} - C_Q) Q = 0$, $\mathfrak{F} \in \cup C_{\mathfrak{R}} - C_Q$, and hence $Q^{\mathfrak{e},\mathfrak{F}} = 0$ by definition.

Now we suppose that $\mathfrak{F} \in \cup C_Q$. By virtue of Theorem 42.2, we can find $S \in \overline{\mathfrak{R}}$ such that

$$S \leqq Q + (C_{\mathfrak{R}} - C_Q), \qquad C_S = C_{\mathfrak{R}},$$

and furthermore $S\mathfrak{R}$ is a simple ring. For such S we have obviously $C_Q S \leqq C_Q Q \leqq Q$, and $C_{[\mathfrak{e}]} C_Q S \mathfrak{R}$ also is a simple ring by Theorem 20.6. On the other hand $[\mathfrak{e}] C_Q \mathfrak{R}$ is a simple ring by Theorem 20.5 and isomorphic to $C_{[\mathfrak{e}]} C_Q S \mathfrak{R}$, because we have by the formula §14(16')

$$C_{C_{[\mathfrak{e}]} C_Q S} = C_{[\mathfrak{e}]} C_Q C_S = C_{[\mathfrak{e}]} C_Q.$$

Therefore we can find by Theorem 41.4 $x \in C_{[\mathfrak{e}]} C_Q S R$ such that

$$\| P C_Q \mathfrak{e} \| = \| P x \| \qquad \text{for every } P \in \mathfrak{R}.$$

Since $\mathfrak{F} \in C_Q$ by assumption, we have hence by definition

$$\| x^{\mathfrak{e},\mathfrak{F}} \| = \| \mathfrak{e}^{\mathfrak{e},\mathfrak{F}} \| = 1.$$

Furthermore we have

$$Q^{\mathfrak{e},\mathfrak{F}} x^{\mathfrak{e},\mathfrak{F}} = (Q x)^{\mathfrak{e},\mathfrak{F}} = (C_Q S x)^{\mathfrak{e},\mathfrak{F}} = x^{\mathfrak{e},\mathfrak{F}},$$

because $Q \geqq C_Q S$ and $C_Q S R \ni x$. Consequently we obtain $Q^{\mathfrak{e},\mathfrak{F}} \neq 0$. Therefore we have

Theorem 54.1. *For a projection operator $Q \in \overline{\mathfrak{R}}$ we have*

$$Q^{\mathfrak{e},\mathfrak{F}} = 0 \quad \text{if and only if} \quad \mathfrak{F} \overline{\in} \cup C_Q.$$

Let $\mathfrak{r}$ be a closed ring of projection operators containing 1, and U a translator of $\mathfrak{r}$. Putting

$$\mathfrak{R} = \{P : \; P U = U P, \; P \in \mathfrak{r}\},$$

we see easily by the formulas (2) and (3) in §41 that $\mathfrak{R}$ also is a closed ring containing $\mathfrak{A}$ and $\mathfrak{R} \subset \mathfrak{r} \subset \overline{\mathfrak{R}}$. Putting

$$\mathfrak{r}^{e,\mathfrak{F}} = \{ Q^{e,\mathfrak{F}} : \ Q \in \mathfrak{r} \},$$

we obtain obviously by (3) a ring of projection operators $\mathfrak{r}^{e,\mathfrak{F}}$ on the ideal space $R^{e,\mathfrak{F}}$ by $\mathfrak{R}$. Furthermore the reduced operator $U^{e,\mathfrak{F}}$ of U is a translator of $\mathfrak{r}^{e,\mathfrak{F}}$. In fact we have by the formulas (2) and (3)

$$U^{e,\mathfrak{F}} Q^{e,\mathfrak{F}} U^{e,\mathfrak{F}*} = (U Q U^{*})^{e,\mathfrak{F}} \in \mathfrak{r}^{e,\mathfrak{F}}.$$

For some $Q \in \mathfrak{r}$, if $0 \neq Q^{e,\mathfrak{F}} \neq 1$, then we have by definition $PQ \overline{\in} \mathfrak{R}$ for $\mathfrak{F} \in U_\rho$, $P \in \mathfrak{R}$, and hence $PQU \neq UPQ$ for $\mathfrak{F} \in U_\rho$, $P \in \mathfrak{R}$. Thus we obtain

$$P(1 - Q) U Q U^{*} \neq 0 \qquad \text{for } \mathfrak{F} \in U_\rho, \ P \in \mathfrak{R}.$$

This relation yields by Theorem 54.1

$$(1 - Q^{e,\mathfrak{F}}) U^{e,\mathfrak{F}} Q^{e,\mathfrak{F}} U^{e,\mathfrak{F}*} \neq 0,$$

and hence $Q^{e,\mathfrak{F}} U^{e,\mathfrak{F}} \neq U^{e,\mathfrak{F}} Q^{e,\mathfrak{F}}$. Therefore we have

Theorem 54.2. *For a translator U of a closed ring of projection operators $\mathfrak{r}$ containing 1, putting*

$$\mathfrak{R} = \{ P : \ UP = PU, \ P \in \mathfrak{r} \},$$

we obtain a closed subring $\mathfrak{R}$ of $\mathfrak{r}$ containing 1, and the reduced operator $U^{e,\mathfrak{F}}$ of U in the ideal space $R^{e,\mathfrak{F}}$ by $\mathfrak{R}$ also is a translator of the ring $\{ Q^{e,\mathfrak{F}} : \ Q \in \mathfrak{r} \}$ such that we have

$$Q^{e,\mathfrak{F}} \ U^{e,\mathfrak{F}} = U^{e,\mathfrak{F}} \ Q^{e,\mathfrak{F}}, \quad Q \in \mathfrak{r},$$

if and only if $Q^{e,\mathfrak{F}} = 1$ or 0.

§55 Measure Preserving Transformations

Let $\mathfrak{m}$ be a totally additive set class in an abstract space Ω and μ a measure on $\mathfrak{m}$. With the same notations in §48, we shall say that a sequence $\varphi_\nu \in L_2$ $(\nu = 1, 2, \ldots)$ is *mean convergent* by μ, if we can find $\varphi \in L_2$ for which

$$\lim_{\nu \to \infty} \int_{\Omega} |\varphi_\nu(\zeta) - \varphi(\zeta)|^2 \mu(d\zeta) = 0.$$

Then we have obviously by definition

Theorem 55.1. *A sequence $\varphi_\nu \in L_2$ $(\nu = 1, 2, \ldots)$ is mean convergent if and only if C_{φ_ν} $(\nu = 1, 2, \ldots)$ is convergent in the (L_2)-*

space R by μ.

Let $E(\mathfrak{E})$ $(\mathfrak{E} \in \mathfrak{m})$ be the spectrality of the (L_2)-space R by μ and $\mathfrak{R}$ the spectral ring of E. For $\varphi, \psi \in L_2$, if we have $C_\varphi \prec C_\psi$ by $\mathfrak{R}$, then we have by definition for every $\mathfrak{E} \in \mathfrak{m}$

$$\int_{\mathfrak{E}} |\varphi(\zeta)|^2 \mu(d\zeta) = \| E(\mathfrak{E}) C_\varphi \|^2 \leqq \| E(\mathfrak{E}) C_\psi \|^2 = \int_{\mathfrak{E}} |\psi(\zeta)|^2 \mu(d\zeta).$$

On the other hand, putting

$$\mathfrak{E}_0 = \{\zeta : |\varphi(\zeta)| > |\psi(\zeta)|\},$$

we have $|\varphi(\zeta)| > |\psi(\zeta)|$ for every $\zeta \in \mathfrak{E}_0$, and hence

$$\int_{\mathfrak{E}_0} |\varphi(\zeta)|^2 \mu(d\zeta) \geqq \int_{\mathfrak{E}_0} |\psi(\zeta)|^2 \mu(d\zeta).$$

Thus we conclude

$$\int_{\mathfrak{E}_0} \{|\varphi(\zeta)|^2 - |\psi(\zeta)|^2\} \mu(d\zeta) = 0,$$

and hence $\mu(\mathfrak{E}_0) = 0$, because $|\varphi(\zeta)| > |\psi(\zeta)|$ for every $\zeta \in \mathfrak{E}_0$. Conversely if $\mu(\mathfrak{E}_0) = 0$, then we see easily by definition that we have $C_\varphi \prec C_\psi$. Therefore we obtain

Theorem 55.2. *We have $C_\varphi \prec C_\psi$ by $\mathfrak{R}$ if and only if*

$$\mu(\{\zeta : |\varphi(\zeta)| \leqq |\psi(\zeta)|\}) = \mu(\Omega).$$

A sequence $\varphi_\nu \in L_2$ $(\nu = 1, 2, \ldots)$ is said to be *convergent almost everywhere* by μ, if we can find $\varphi \in L_2$ and $\mathfrak{E} \in \mathfrak{m}$ such that we have $\mu(\mathfrak{E}) = \mu(\Omega)$ and

$$\lim_{\nu \to \infty} \varphi_\nu(\zeta) = \varphi(\zeta) \qquad \text{for every } \zeta \in \mathfrak{E}.$$

With this definition we have

Theorem 55.3. *A sequence $\varphi_\nu \in L_2$ $(\nu = 1, 2, \ldots)$ is convergent almost everywhere by μ, if and only if we can find a sequence $\mathfrak{R} \ni P_\rho \uparrow_{\rho=1}^{\infty} 1$ and an element $\alpha \in R$ such that*

$$(\mathfrak{R})\text{-}\lim_{\nu \to \infty} P_\rho C_{\varphi_\nu} = P_\rho \alpha \qquad \text{for every } \rho = 1, 2, \ldots.$$

Proof. If $(\mathfrak{R})\text{-}\lim_{\nu \to \infty} C_{\varphi_\nu} = 0$, then there is by definition a function $\psi \in L_2$ such that for any $\varepsilon > 0$ we can find ν_0 for which $C_{\varphi_\nu} \prec \varepsilon C_\psi$ for every $\nu \geqq \nu_0$. Thus, putting

$$\mathfrak{E} = \prod_{\mu=1}^{\infty} \sum_{\rho=1}^{\infty} \prod_{\nu=\rho}^{\infty} \{ \zeta : |\varphi_\nu(\zeta)| \leqq \frac{1}{\mu} |\Psi(\zeta)| \},$$

we see easily by Theorem 55.2 that $\mu(\mathfrak{E}) = \mu(\Omega)$ and

$$\lim_{\nu \to \infty} \varphi_\nu(\zeta) = 0 \qquad \text{for every } \zeta \in \mathfrak{E}.$$

For a sequence $\mathcal{R} \ni P_\rho \uparrow_{\rho=1}^{\infty} 1$ we can find by Theorem 48.3 a sequence $\mathfrak{E}_\rho \in \mathfrak{M}$ such that $P_\rho = E(\mathfrak{E}_\rho)$ $(\rho = 1, 2, \ldots)$. For such $\mathfrak{E}_\rho$ $(\rho = 1, 2, \ldots)$ we have $E(\sum_{\rho=1}^{\infty} \mathfrak{E}_\rho) = 1$, and hence by Theorem 48.2

$$\mu(\sum_{\rho=1}^{\infty} \mathfrak{E}_\rho) = \mu(\Omega).$$

If $(\mathcal{R})\text{-}\lim_{\nu \to \infty} P_\rho C_{\varphi_\nu} = 0$ for every $\rho = 1, 2, \ldots$, then, since

$$P_\rho C_{\varphi_\nu} = E(\mathfrak{E}_\rho) C_{\varphi_\nu} = C_{\chi_{\mathfrak{E}_\rho} \varphi_\nu},$$

we can find $\mathfrak{F}_\rho \in \mathfrak{M}$ such that $\mu(\mathfrak{F}_\rho) = \mu(\Omega)$ and

$$\lim_{\nu \to \infty} \chi_{\mathfrak{E}_\rho}(\zeta) \varphi_\nu(\zeta) = 0 \qquad \text{for every } \zeta \in \mathfrak{F}_\rho, \ \rho = 1, 2, \ldots$$

as proved just above. Then we have

$$\lim_{\nu \to \infty} \varphi_\nu(\zeta) = 0 \qquad \text{for every } \zeta \in \sum_{\rho=1}^{\infty} \mathfrak{E}_\rho \mathfrak{F}_\rho,$$

$$\mu(\sum_{\rho=1}^{\infty} \mathfrak{E}_\rho \mathfrak{F}_\rho) \geqq \mu((\sum_{\rho=1}^{\infty} \mathfrak{E}_\rho)(\prod_{\rho=1}^{\infty} \mathfrak{F}_\rho)) = \mu(\Omega),$$

and hence φ_ν $(\nu = 1, 2, \ldots)$ is convergent almost everywhere by μ.

Conversely if $\varphi_\nu \in L_2$ $(\nu = 1, 2, \ldots)$ is convergent to 0 for $\zeta \in \mathfrak{F}$, $\mu(\mathfrak{F}) = \mu(\Omega)$, then, putting for $\rho = 1, 2, \ldots$

$$\mathfrak{F}_\rho = \{ \zeta : |\varphi_\nu(\zeta)| \leqq \rho \text{ for every } \nu = 1, 2, \ldots \} \mathfrak{F},$$

$$\Psi_{\rho,\nu}(\zeta) = \sup_{\sigma \geqq \nu} |\chi_{\mathfrak{F}_\rho}(\zeta) \varphi_\sigma(\zeta)|,$$

we see easily that $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \ldots$, $\mu(\sum_{\rho=1}^{\infty} \mathfrak{F}_\rho) = \mu(\Omega)$,

$$\Psi_{\rho,1}(\zeta) \geqq \Psi_{\rho,2}(\zeta) \geqq \ldots, \quad \lim_{\nu \to \infty} \Psi_{\rho,\nu}(\zeta) = 0 \quad \text{for } \zeta \in \Omega,$$

$$|\chi_{\mathfrak{F}_\rho}(\zeta) \varphi_\nu(\zeta)| \leqq \Psi_{\rho,\nu}(\zeta) \qquad \text{for } \zeta \in \Omega, \ \rho, \nu = 1, 2, \ldots.$$

Thus we obtain $E(\mathfrak{F}_\rho) \uparrow_{\rho=1}^{\infty} 1$ by Theorem 48.2,

$$C_{\Psi_{\rho,1}} \succ C_{\Psi_{\rho,2}} \succ \ldots, \quad \lim_{\nu \to \infty} C_{\Psi_{\rho,\nu}} = 0 \quad (\rho = 1, 2, \ldots)$$

by Theorem 55.2 and by Lemma 2 in §48, and further

$$E(\mathfrak{F}_\rho) C_{\varphi_\nu} = C_{\chi_{\mathfrak{F}_\rho} \varphi_\nu} \prec C_{\Psi_{\rho,\nu}} \qquad (\rho, \nu = 1, 2, \ldots)$$

by Theorem 55.2. Therefore we conclude by Theorem 21.7

$$(\mathcal{R})\text{-}\lim_{\nu \to \infty} E(\mathfrak{F}_\rho) C_{\varphi_\nu} = 0 \qquad \text{for every } \rho = 1, 2, \ldots.$$

A one-to-one correspondence $T\zeta \in \Omega$ $(\zeta \in \Omega)$ from Ω onto Ω itself is called a <u>transformation</u> of $\mathfrak{M}$, if $T\mathfrak{M} = \mathfrak{M}$, making use of the notations:

$$T\mathfrak{E} = \{ T\zeta : \zeta \in \mathfrak{E} \}, \qquad T\mathfrak{M} = \{ T\mathfrak{E} : \mathfrak{E} \in \mathfrak{M} \}.$$

For a transformation T of $\mathfrak{M}$, using the notation:

$$\varphi T(\zeta) = \varphi(T\zeta) \qquad (\zeta \in \Omega)$$

for functions φ on Ω, we see easily that φT is $(\mathfrak{M})$-measurable if and only if φ is so.

For a transformation T of $\mathfrak{M}$ we shall say that T preserves the measure μ, if $\mu(T\mathfrak{E}) = \mu(\mathfrak{E})$ for every $\mathfrak{E} \in \mathfrak{M}$.

Let T be a transformation of $\mathfrak{M}$ which preserves the measure μ in the sequel. Putting

$$U_T C_\varphi = C_{\varphi T} \qquad (\varphi \in L_2),$$

we see easily that U_T is a linear operator on the (L_2)-space R by μ. Since we have for every $\varphi \in L_2$

$$\| U_T C_\varphi \|^2 = \int_\Omega |\varphi(T\zeta)|\mu(d\zeta) = \int_\Omega |\varphi(T\zeta)|\mu(dT\zeta) = \| C_\varphi \|^2,$$

and the range of U_T coincides with R, U_T is a unitary operator on R. Thus, recallin Theorem 55.1, we obtain by Theorem 50.1

Theorem 55.4. *For every transformation T of $\mathfrak{M}$, if T preserves a measure μ on $\mathfrak{M}$, then $\frac{1}{\nu}\sum_{\rho=0}^{\nu-1} \varphi T^\rho$ $(\nu = 1, 2, \ldots)$ is mean convergent by μ for every $\varphi \in L_2$.*

This Theorem 55.4 is called Mean Ergodic Theorem.

For the characteristic function $\chi_{\mathfrak{E}}$ of $\mathfrak{E} \in \mathfrak{M}$ we have obviously

$$\chi_{\mathfrak{E}}(T\zeta) = \chi_{T^{-1}\mathfrak{E}}(\zeta) \qquad (\zeta \in \Omega)$$

for the inverse transformation T^{-1} of T, because $\mathfrak{E} \ni T\zeta$ if and only if $T^{-1}\mathfrak{E} \ni \zeta$. Thus we have by definition for every $\varphi \in L_2$

$$U_T E(\mathfrak{E}) C_\varphi = C_{(\chi_{\mathfrak{E}} T)(\varphi T)} = C_{\chi_{T^{-1}\mathfrak{E}}(\varphi T)} = E(T^{-1}\mathfrak{E}) U_T C_\varphi,$$

and hence $U_T E(\mathfrak{E}) U_T^* = E(T^{-1}\mathfrak{E})$. Accordingly U_T is a translator of the spectral ring $\mathfrak{R}$ of E. Therefore, recalling Theorem 55.3, we obtain by Theorem 52.4

Theorem 55.5. *For every transformation T of $\mathfrak{M}$, if T preserves a measure μ on $\mathfrak{M}$, then $\frac{1}{\nu}\sum_{\rho=0}^{\nu-1} \varphi T^\rho$ $(\nu = 1, 2, \ldots)$ is convergent almost everywhere by μ.*

This Theorem 55.5 is called Individual Ergodic Theorem.

APPENDIX PROOF OF $\mathfrak{k}\mathfrak{k} = \mathfrak{k}$

In Proof of Theorem 43.3 we have admitted without proof the fact that for a system of different cardinal numbers $\mathfrak{k}_\lambda \leqq \mathfrak{k}$ $(\lambda \in \Lambda)$ we have $\sum_{\lambda \in \Lambda} \mathfrak{k}_\lambda \leqq \mathfrak{k}$, if $\mathfrak{k}$ is infinite. Since the density of $\mathfrak{k}_\lambda$ is not greater than $\mathfrak{k}$, this fact follows immediately from

Theorem $\mathfrak{k}\mathfrak{k} = \mathfrak{k}$ for every infinite cardinal number $\mathfrak{k}$.

Proof. Let Ω be a space with infinite density. For every set $A \neq 0$ we denote by A^ω the totality of finite systems of elements from A, that is,

$$A^\omega = \{(a_1, a_2, \ldots, a_\rho) : a_\nu \in A \ (\nu = 1, \ldots, \rho), \ \rho = 1, 2, \ldots\}$$

With this notation it is evident that for every A and B with same density the density of A^ω coincides with that of B^ω. Furthermore the product of the densities of A^ω and B^ω is obviously not greater than the density of $(A + B)^\omega$. Thus, denoting by $\mathfrak{k}_A$ the density of A^ω, if the density of A is infinite, then $\mathfrak{k}_A \mathfrak{k}_A = \mathfrak{k}_A$.

Therefore we need only prove that there exists a subset $A \subset \Omega$ for which the density of Ω coincides with that of A^ω. By virtue of Maximal Theorem we can find a maximal subset $A \subset \Omega$ with a one-to-one correspondence between A^ω and a subset $B_A \subset \Omega$ such that $A \subset B_A$ and the density of A is infinite. Then the density of $\Omega - B_A$ is not greater than that of A^ω. Because, if the density of $\Omega - B_A$ is greater than that of A^ω, then for an element $a \in \Omega - B_A$ the density of $\{a, A\}^\omega - A^\omega$ is less than that of $\Omega - B_A$, since the density of $\{a, A\}^\omega$ coincides with that of A^ω. Thus we can find a subset $B_{\{a,A\}} \subset \Omega$ and a one-to-one correspondence between $\{a, A\}^\omega$ and $B_{\{a,A\}}$ such that $\{a, A\} \subset B_{\{a,A\}}$ and this correspondence includes the correspondence between A and B_A, contradicting the maximality of the subset A. Therefore the density of $\Omega - B_A$ is not greater than that of A^ω, and hence the density of Ω coincides with that of A^ω, because the density of Ω is infinite.

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The Hilbert space with countable dimension was abstractly defined and discussed first by J. von Neumann (1). Generalization to arbitrary dimension was made by H. Löwig (1) and independently by F. Rellich (1). Theorem 4.1 was obtained by P. Jordan and J. von Neumann (1).

Properties of projection operators was systematically discussed first by J. von Neumann (1). Theorem 12.8 was published first by H. Nakano in Japanese: On a theorem in the Hilbert space, Zenkoku Sūgaku Danwakai, Osaka 192(1940) 39-42. Properties about systems of projection operators was obtained by H. Nakano (2), (3). Rings of projection operators was considered first by H. Nakano (2).

The concept of relative spectra was gived by H. Nakano (8) in semi-ordered linear spaces. The definition of relative spectra in the text was given by H. Nakano (9) as the relative spectra of continuous linear functionals on semi-ordered linear spaces. But, in semi-ordered linear spaces, we can consider only relative spectra for an axis.

The notion of dilatators in the text is a generalization of that defined by H. Nakano (7) in semi-ordered linear spaces, which corresponds to the dilatators for an axis in the text. Functions of dilatators discussed in §30 correspond to functions of self-adjoint operators discussed first by J. von Neumann (3). In the latter form, Theorem 30.7 was obtained by J. von Neumann (3).

Normal operators defined by J. von Neumann (2) coincides essentially with regular closed normal operators in the text. The definition of normal operators in the text was gived by H. Nakano (1). The concept of regularity of normal operators was obtained by H. Nakano (1). Spectralization of normal operators in the text §34 is due to H. Nakano (1). An elementary proof of Theorem 33.5 was obtained first by B. A. Lengyel and M. H. Stone (1).

Spectralization of self-adjoint operators was considered first by J. von Neumann (1), and then by F. Riesz (1), and independently by M. H. Stone (1). The definition of self-adjoint operators in the text is due to M. H. Stone (1).

Graphs of operators was considered first by J. von Neumann (4). Theorems 36.5, 39.1, and 39.4 wre obtained by J. von Neumann. The proof of Theorem 36.5 in the text is due to H. Nakano (4). The definition of Hermitean operators in the text was given by J. von Neumann (1) and the theory explained in the text §40 is due to J. von Neumann (1).

The theory developed in Chapter V is due to H. Nakano (5), (6). P. R. Halmos (1) explained this theory by means of measures instead of regular σ-ideals.

As a generalization of the mean ergodic theorem proved first by J. von Neumann (5), the mean ergodic theorems of bounded linear operators were discussed by many mathematicians in Banach spaces. The proof of Theorem 50.1 is due to G. Birkhoff (1). The ergodic theorems of dilatators explained in §51 were obtained by H. Nakano (7). The ergodic theorems of translators explained in §52 were discussed by H. Nakano (10) in semi-ordered linear spaces. Individual ergodic theorem explained in §55 was proved first by G. D. Birkhoff (1). Reduction theory was considered first by J. von Neumann (6). But the theory discussed in §53 is different essentially from that of J. von Neumann (6).

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